Zetas and moments of finite group actions

By Kazufumi KIMOTO*)

Graduate School of Mathematics, Kyushu University, 6-10-1, Hakozaki, Higashi-ku, Fukuoka 812-8581

(Communicated by Heisuke HIRONAKA, M. J. A., Oct. 14, 2003)

Abstract: We introduce and study two kinds of zeta functions $\zeta(u;G,X)$ and $Z(u;G,X)$ as well as moments $m(k;G,X)$ attached to a given finite group action $G \curvearrowright X$. We show that zeta functions determine the moments, and moments determine the multiple transitivity of the action. In the symmetric group case we give an explicit formula of moments and calculate zeta functions of the infinite symmetric group $\mathfrak{S}_\infty$.

Key words: Zeta functions; finite group actions.

1. Introduction and preliminaries. In this paper we introduce and study two kinds of zeta functions attached to finite group actions.

We first recall the definition of a zeta function attached to a shift dynamical system [5]. Let X be a finite set. Denote by $X^{\mathbf{Z}}$ the set

$$X^{\mathbf{Z}} := \{(x_n)_{n\in\mathbf{Z}} \mid x_n \in X\}$$

of all sequences in X indexed by $\mathbf{Z}$. The shift operator Δ on $X^{\mathbf{Z}}$ is a map defined by

$$X^{\mathbf{Z}} \ni \boldsymbol{x} = (x_n)_{n\in\mathbf{Z}} \mapsto \Delta(\boldsymbol{x}) = (x_{n-1})_{n\in\mathbf{Z}} \in X^{\mathbf{Z}}.$$

The pairing $(\Delta, X^{\mathbf{Z}})$ defines a discrete dynamical system called a **full shift**. For arbitrary Δ-invariant subset S of $X^{\mathbf{Z}}$, the pairing (Δ_S, S) also defines a dynamical system called a **subshift** where Δ_S is the restriction of Δ on S.

Assume that an equivalence relation $\sim$ on $X^{\mathbf{Z}}$ is given. We further suppose that the relation $\sim$ is compatible with the shift operator Δ, that is, $\boldsymbol{x} \sim \boldsymbol{y}$ implies $\Delta(\boldsymbol{x}) \sim \Delta(\boldsymbol{y})$. Then, for any given subshift (Δ_S, S), we can define a dynamical system $(\Delta_{\mathcal{O}}, \mathcal{O})$ on a quotient set $\mathcal{O} = S/\sim$. We call this a **quotient shift**. When $\sim$ is the equality '=', a quotient shift is nothing but a subshift itself.

Let $(\Delta_{\mathcal{O}}, \mathcal{O})$ be a quotient shift of $(\Delta, X^{\mathbf{Z}})$. Denote by $\mathrm{fix}(f)$ the number of fixed points

$$\mathrm{fix}(f) := \#\{\boldsymbol{y} \in Y \mid f(\boldsymbol{y}) = \boldsymbol{y}\}$$

of a map $f : \mathcal{O} \to \mathcal{O}$. The **zeta function** of a quotient shift $(\Delta_{\mathcal{O}}, \mathcal{O})$ is defined by

$$\zeta(u;\mathcal{O}) := \exp\left(\sum_{k=1}^{\infty} \mathrm{fix}(\Delta_{\mathcal{O}}^k)\frac{u^k}{k}\right).$$

Group actions and induced dynamical systems. Let G be a finite group acting on a finite set X. We introduce two kinds of zeta fnctions attached to the action $G \curvearrowright X$. These are defined as zeta functions attached to quotient shifts of $X^{\mathbf{Z}}$ which reflect the existence of a group action as we see in the following.

First one is a subshift $(\Delta_{\mathcal{C}(G,X)}, \mathcal{C}(G,X))$ defined by

$$\mathcal{C}(G,X) := \{(g^n x)_{n\in\mathbf{Z}} \in X^{\mathbf{Z}} \mid g \in G, x \in X\}.$$

Namely, this subshift $(\Delta_{\mathcal{C}(G,X)}, \mathcal{C}(G,X))$ is consisting of elements which have periodicity coming from the group action. We denote by $\zeta(u;G,X)$ the attached zeta function, that is, $\zeta(u;G,X) := \zeta(u;\mathcal{C}(G,X))$.

Second one is a quotient shift $\mathcal{O}_{G,X} = X^{\mathbf{Z}}/\sim$ of the full shift $X^{\mathbf{Z}}$, where the relation $\sim$ on $X^{\mathbf{Z}}$ is defined by

$$\boldsymbol{x} \sim \boldsymbol{y} \overset{\mathrm{def}}{\iff} \boldsymbol{x} = g.\boldsymbol{y} \quad (\exists g \in G).$$

It is easy to see that this relation $\sim$ is compatible with Δ. We denote by $Z(u;G,X)$ the attached zeta function, that is, $Z(u;G,X) := \zeta(u;\mathcal{O}_{G,X})$. In this case it is better to deal with a modified log-derivative

$$\Xi(u;G,X) := \frac{1}{u^2}\frac{Z'}{Z}\left(-\frac{1}{u};G,X\right)$$

instead of the zeta function $Z(u;G,X)$ itself.

2000 Mathematics Subject Classification. Primary 16W22; Secondary 11M99.

*) Contact address: Department of Mathematics, University of the Ryukyus, 1 Senbaru, Nishihara, Okinawa 903-0231.

Moments and moment generating functions. Assume that a finite group action $G \curvearrowright X$ on a finite set is given. We put $\mathrm{Per}_r(g; G, X) := \{x \in X \mid g^r x = g,\ g^j x \neq x\ (0 < j < r)\}$, and we denote by $\mathrm{per}_r(g; G, X) := \#\,\mathrm{Per}_r(g; G, X)$ the number of strictly r-periodic points of g. We also put $\mathrm{fix}(g; G, X) := \mathrm{per}_1(g; G, X)$, the number of fixed points of g.

We define the k-th **moment** and k-th **factorial moment** of level r for an action $G \curvearrowright X$ by

$$m_r(k; G, X) := \frac{1}{\#G} \sum_{g \in G} \mathrm{per}_r(g; G \curvearrowright X)^k,$$

$$\mathfrak{m}_r(k; G, X) := \frac{1}{\#G} \sum_{g \in G} \mathrm{per}_r(g; G \curvearrowright X)^{\underline{k}},$$

for $k \in \mathbf{N}$. Here we put $a^{\underline{k}} := a(a-1)\cdots(a-k+1)$. It is convenient to put $m_r(0; G, X) = \mathfrak{m}_r(0; G, X) = 1$. When $r = 1$, we often omit the suffix and simply write as $m(k; G, X)$. We remark that $\mathfrak{m}_r(k; G, X) = 0$ whenever $kr > \#X$. We introduce the following generating function

$$\mathcal{M}_r(z; G, X) := \sum_{g \in G} \frac{(1+z)^{\mathrm{per}_r(g)}}{\#G},$$

and we call this the **moment polynomial**.

For abbreviation we will omit the symbols G and X when they are obviously specified in the context.

2. Euler product.

In this section we show several typical properties of zeta functions defined as above.

First we see that the zeta function $\zeta(u; G, X)$ of a subshift $(\Delta_{\mathcal{C}(G,X)}, \mathcal{C}(G, X))$ has an Euler product expression and determinant expression as expected. First we prepare several conventions.

Definition 2.1. Let $G \curvearrowright X$ be a given group action.

(1) A finite sequence $c = (x_j)_{j=0}^{l-1}$ in X is called a **closed path** if there is some element $g \in G$ such that c is invariant up to cyclic permutations under the action of g. A closed path c is called **prime** if c consists of distinct members.

(2) Two closed paths are defined to be equivalent if and only if they coincide up to cyclic permutation. An equivalence class C of this relation is called a **closed geodesic** in X. A closed geodesic C is called **prime** if a representative of C is prime.

Denote by $\mathrm{Prim}(X)$ the set of all primitive geodesics on X. We also denote by $l(C)$ the number of entries of a representative $c \in C$ and call the **length** of C. Under these conventions, the Euler product expression of a zeta function $\zeta(u; G, X)$ is given as follows:

Theorem 2.1 (Euler product expression)**.** *The zeta function $\zeta(u; G, X)$ for a given finite group action $G \curvearrowright X$ has the following expression.*

$$\begin{aligned}\zeta(u; G, X) &= \prod_{C \in \mathrm{Prim}(X)} (1 - u^{l(C)})^{-1} \\ &= \prod_{l=1}^{\infty} (1 - u^l)^{-Q(l)/l}.\end{aligned} \tag{2.1}$$

Here we denote by $Q(l) = Q(l; G, X)$ the number of prime paths with length l.

Proof. Since the number of elements in $\mathcal{C}(G, X)$ of a minimal period d is also given by $Q(d)$, we have

$$\#\mathcal{C}_k(G, X) = \sum_{d \mid k} Q(d).$$

Thus it follows that

$$\begin{aligned}\zeta(u; G, X) &= \exp\left(\sum_{k=1}^{\infty} \#\mathcal{C}_k(G, X) \frac{u^k}{k}\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \Big(\sum_{d \mid k} Q(d)\Big) \frac{u^k}{k}\right) \\ &= \exp\left(\sum_{l=1}^{\infty} \sum_{k=1}^{\infty} Q(l) \frac{u^{lk}}{lk}\right) \\ &= \prod_{l=1}^{\infty} (1 - u^l)^{-Q(l)/l},\end{aligned}$$

which is the desired expression. Here we use the formula

$$\sum_{k=1}^{\infty} \sum_{\substack{w \in W \\ L(w) \mid k}} f(k) = \sum_{w \in W} \sum_{d=1}^{\infty} f(L(w)d)$$

for a countable set W and a map $L : W \to \mathbf{N}$. □

Remark 2.1. The zeta function $Z(u; G, X)$ does **not** have such an Euler product expression because $Z(u; G, X)$ is defined for a (non-trivial) quotient shift, not a subshift. In fact, if the relation $\sim$ is non-trivial, the manipulation used in the proof above does not work.

Denote by $L(X)$ the $\mathbf{C}$-linear space consisting of $\mathbf{C}$-valued functions on a set X. If X is a G-set, $L(X)$ naturally gives a representation of G. The shift operator $\Delta_{\mathcal{C}(G,X)}$ extends to a linear transformation on $L(\mathcal{C}(G,X))$, which is compatible with the action of G.

Theorem 2.2 (Determinant expression)**.** *The zeta function has a determinant expression*

$$\zeta(u;G,X) = \det(1-u\Delta_{\mathcal{C}(G,X)})^{-1}. \tag{2.2}$$

Proof. This is immediate from the Euler product expression and the following elementary formula

$$\begin{vmatrix} 1 & & & & -u \\ -u & 1 & & & \\ & -u & 1 & & \\ & & \ddots & \ddots & \\ & & & -u & 1 \end{vmatrix} = 1-u^n.$$

□

3. Zeta functions and moments. Next we see the relation between zeta functions and moments. As we see in the following, the zeta function $\zeta(u;G,X)$ essentially determines the multiple transitivity of attached action $G \curvearrowright X$ in view of Theorem 3.4.

Proposition 3.1. *For any given action $G \curvearrowright X$, we have*

$$Q(l;G,X) = \sum_{g\in G}\sum_{x\in X} \frac{1}{\#G_x(g)}\delta(g;l,x). \tag{3.1}$$

Here we put

$$\delta(g;l,x) = \begin{cases} 1 & x \in \mathrm{Per}_l(g;G,X), \\ 0 & \textit{otherwise}, \end{cases}$$

$$G_x(g) = \bigcap_{j=0}^{\infty} G_{g^j x}$$

for $g \in G$, $x \in X$, $l \in \mathbf{N}$.

Proof. Two prime paths $(x, gx, \ldots)$ and $(x, hx, \ldots)$ are equal if and only if $g^j x = h^j x$ ($\Longleftrightarrow$ $g^{-j}h^j \in G_x$) ($j \geq 1$). Since

$$g^j x = h^j x = h(h^{j-1}x) = h(g^{j-1}x)$$
$$\Longleftrightarrow g^{-1}h(g^{j-1}x) = g^{j-1}x,$$

every $g^j x$ is a fixed point of $g^{-1}h$. This implies that $g^{-1}h \in G_x(g) = \bigcap_{j=0}^{\infty} G_{g^j x}$, which is equivalent to the equality $G_x(g) = G_x(h)$. Thus we have

$$Q(l;G,X) = \sum_{x\in X}\sum_{g\in G} \frac{1}{\#G_x(g)}\delta(g;l,x).$$

This completes the proof. □

Theorem 3.2. *A group action $G \curvearrowright X$ is k-transitive only if $m_k(1;G,X) = 1$.*

Proof. Assume $G \curvearrowright X$ is k-transitive. Denote by $X^{\underline{k}}$ the subset of X^k consisting of elements with distinct entries. When $\delta(k;g,x) = 1$, the sequence $\{x, gx, \ldots, g^{k-1}x\}$ consists of distinct k members, which implies that the subgroup $G_x(g)$ is nothing but the stabilizer of the element $(x, gx, \ldots, g^{k-1}x) \in X^{\underline{k}}$ under the induced action $G \curvearrowright X^{\underline{k}}$, which is transitive by assumption of k-transitivity. Therefore we have

$$X^{\underline{k}} \cong G/G_x(g) \implies \#X^{\underline{k}} = \frac{\#G}{\#G_x(g)}. \tag{3.2}$$

We also notice that $Q(k;G,X) = \#X^{\underline{k}}$ if $G \curvearrowright X$ is k-transitive. In fact, for any sequence $c = (x_1, \ldots, x_k)$ consisting of distinct k members, there exists an element $g \in G$ such that $gx_i = x_{i+1}$ $(1 \leq i < k)$, $gx_k = x_1$, which means $Q(k;G,X) = (\#X)^{\underline{k}} = \#X^{\underline{k}}$.

Since

$$\mathrm{per}_k(g;G,X) = \sum_{x\in X}\delta(k;g,x),$$

we have

$$\begin{aligned} &m_k(1;G,X) - 1 \\ &= \frac{1}{\#G}\sum_{g\in G}\mathrm{per}_k(g;G,X) - \frac{1}{\#X^{\underline{k}}}Q(k;G,X) \\ &= \frac{1}{\#X^{\underline{k}}}\sum_{g\in G}\sum_{x\in X}\left(\frac{\#X^{\underline{k}}}{\#G} - \frac{1}{\#G_x(g)}\right)\delta(k;g,x), \end{aligned}$$

which is 0 by (3.2). □

Another zeta function $Z(u;G,X)$ is directly connected with the multiple transitivity of attached action. Actually, $Z(u;G,X)$ has a following expression as a generating function of moments.

Theorem 3.3 ([2])**.** *We have*

$$Z(u;G,X) = \exp\left(\sum_{k=1}^{\infty} m(k)\frac{u^k}{k}\right). \tag{3.3}$$

From the expression (3.3) of $Z(u;G,X)$, the modified log-derivative $\Xi(u;G,X)$ is calculated as follows:

$$\begin{aligned} \Xi(u;G,X) &= \frac{1}{u} - \frac{1}{\#G}\sum_{g\in G}\frac{1}{u+\mathrm{fix}(g)} \\ &= -\frac{1}{u}\sum_{j=1}^{\infty}\mathfrak{m}(j)\frac{(-1)^j}{(u+1)_j}. \end{aligned}$$

Here we denote by $(a)_k := a(a+1)\cdots(a+k-1)$ the Pochhammer's symbol.

The moments are used to describe criteria of multiple transitivity of finite group actions.

Theorem 3.4 ([2]). *Suppose that a finite group action $G \curvearrowright X$ is given, Then the following conditions are equivalent for every $k \leq \#X$.*

(a) $G \curvearrowright X$ *is k-transitive.*
(b) $G \curvearrowright X^{\underline{k}}$ *is transitive.*
(c) $m(k; G, X) = \beta_k$.
(d) $\mathfrak{m}(k; G, X) = 1$.
(e) $\mathcal{M}^{(k)}(0) = 1$.

*Here we denote by $\beta_k := \sum_{j=0}^{k} \left\{ {k \atop j} \right\}$ the k-th **Bell number**.*

4. Zeta functions of abelian group actions. In general, the zeta function of the action $G \curvearrowright G$ is given by

$$\zeta(u; G, G) = \prod_{l=1}^{\infty} (1 - u^l)^{-\frac{\psi(l,G)}{l}\#G} \tag{4.1}$$

where $\psi(l, G)$ denotes the number of elements in G with order l. In fact, for any $g_0 \in G$, two paths $(g_0, gg_0, \ldots)$ and $(g_0, g'g_0, \ldots)$ are different if $g \neq g'$, and their length are given by the order of g, g'. When G is abelian, we can determine $\psi(l, G)$ by the fundamental theorem of finite abelian groups.

Lemma 4.1. *If G is a finite abelian group of type $(\alpha_1, \alpha_2, \ldots, \alpha_k)$, then we have*

$$\psi(d, G) = \sum_{\substack{r_1|\alpha_1, r_2|\alpha_2, \ldots, r_k|\alpha_k \\ \mathrm{lcm}(r_1, r_2, \ldots, r_k) = d}} \varphi(r_1)\varphi(r_2)\ldots\varphi(r_k).$$

Here 'lcm' *stands for the least common multiple.*

Proof. When $g = (g_1, \ldots, g_k) \in G$ is of order d, we have $g_j^d = 1$ for each j, that is, the order of g_j divides d. Conversely, if the order of g_j is equal to r_j, the order of $g = (g_1, \ldots, g_k)$ is equal to $\mathrm{lcm}(r_1, r_2, \ldots, r_k)$. $\square$

For a general transitive action $G \curvearrowright X \cong G/H$ of a finite abelian group G, we see that $\mathrm{Ker}(G, X) = H$. Since it is clear that $\mathcal{C}(G, X) = \mathcal{C}(G/H, X)$, we notice

$$\zeta(u; G, X) = \zeta(u; G/H, G/H).$$

Thus all zeta functions of abelian group actions are determined, and we have the

Theorem 4.2. *Let $G_j \curvearrowright X_j$ $(j = 1, 2)$ be transitive actions of finite abelian groups. Then we have*

$$\begin{aligned} &\zeta(u; G_1, X_1) = \zeta(u; G_2, X_2) \\ &\iff G_1/\mathrm{Ker}(G_1, X_1) \cong G_2/\mathrm{Ker}(G_2, X_2). \end{aligned} \tag{4.2}$$

Remark 4.1. This result is not true in general if G is not abelian.

Remark 4.2. The zeta function $Z(u; G, X)$ attached to a transitive action $G \curvearrowright X$ of a finite abelian group G is given by

$$Z(u; G, X) = (1 - u\#X)^{-1/\#X}.$$

In fact, it is easy to see that $m(k; G, X) = (\#X)^{k-1}$. Hence, this zeta function $Z(u; G, X)$ cannot distinguish the group actions even in the case of abelian groups.

5. Calculation of infinite symmetric group case. If we think of the action $\mathfrak{S}_\infty \curvearrowright [\infty] = \mathbf{N}$, the definitions of the zeta functions $\zeta(u; \mathfrak{S}_\infty, [\infty])$ and $Z(u; \mathfrak{S}_\infty, [\infty])$ do not make sense. Therefore, if we want to deal with such a case, we need some modification. One way to obtain a zeta function of $\mathfrak{S}_\infty$ is to take a normalized limit of zeta functions for finite symmetric groups $\mathfrak{S}_n$.

The zeta function $\zeta(u; \mathfrak{S}_n, [n])$ of $\mathfrak{S}_n \curvearrowright [n] = \{1, 2, \ldots, n\}$ is given by

$$\zeta(u; \mathfrak{S}_n, [n]) = \prod_{k=1}^{n} (1 - u^k)^{-n^{\underline{k}}/k}. \tag{5.1}$$

In fact, every distinct k-members $(x_1, \ldots, x_k)$ in $[n]$ realize a prime geodesic on $[n]$, which means that the number of prime geodesics of length k is given by $(1/k)\binom{n}{k} = (n^{\underline{k}}/k)$. It follows that

$$\begin{aligned} \zeta(u/n; \mathfrak{S}_n, [n]) &= \prod_{k=1}^{n} \left(1 - \frac{u^k}{n^k}\right)^{-n^{\underline{k}}/k} \\ &\xrightarrow{n \to \infty} \prod_{k=1}^{\infty} \exp\left(\frac{u^k}{k}\right) = (1 - u)^{-1}, \end{aligned}$$

which is not very interesting.

On the other hand, the function $\Xi(u; \mathfrak{S}_n, [n])$ has a limit when $n \to \infty$ as follows:

Theorem 5.1. *The sequence $\{\Xi(u; \mathfrak{S}_n, [n])\}_{n=1}^{\infty}$ converges absolutely and uniformly on any compact domain in $\mathbf{C}$. The limit function*

$$\Xi(u; \mathfrak{S}_\infty) := \lim_{n \to \infty} \Xi(u; \mathfrak{S}_n, [n])$$

have the expression

$$\Xi(u; \mathfrak{S}_\infty) = \frac{1 - {}_1F_1(1; u + 1; -1)}{u}. \tag{5.2}$$

Here ${}_1F_1(a, c; z)$ is the confluent hypergeometric function of Kummer's type

$$ {}_1F_1(a;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}. $$

For the proof and further properties, see [2].

6. Moments of symmetric groups. We recall briefly the basic notions on the symmetric groups.

Conjugacy classes of the symmetric group $\mathfrak{S}_n$ are labeled by the **cycle type**. An element in $\mathfrak{S}_n$ is said to be of type $1^{p_1}2^{p_2}\cdots n^{p_n}$ when it is decomposed into a disjoint product of p_1 1-cycles, p_2 2-cycles, ..., and p_n n-cycles. Two elements in $\mathfrak{S}_n$ are conjugate if and only if they are of the same type. We also denote by $1^{p_1}2^{p_2}\cdots n^{p_n}$ the corresponding conjugacy class. The set of all conjugacy classes of $\mathfrak{S}_n$ is denoted by $\mathrm{Conj}(\mathfrak{S}_n)$. It is clear by definition that

$$ \mathrm{per}_r(\sigma;n) = rp_r \quad \text{if} \quad \sigma \in 1^{p_1}2^{p_2}\cdots n^{p_n}. $$

In particular, the function $\mathrm{per}_r(\,\cdot\,;n)$ is a **class function**.

Irreducible representations of $\mathfrak{S}_n$ are parametrized by **Young diagrams** with n boxes. The irreducible character corresponding to a diagram λ is denoted by χ^λ. We denote the trivial character of $\mathfrak{S}_n$ by $\mathbf{1}_n$. Let $\mathbf{Z}(\mathfrak{S}_n)$ be the set of integral linear combinations of irreducible characters of $\mathfrak{S}_n$, which is a subring of the group algebra $\mathbf{C}[\mathfrak{S}_n]$. Recall that $\mathbf{C}[\mathfrak{S}_n]$ possesses a canonical invariant inner product

$$ \langle f_1,\, f_2\rangle_n := \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} f_1(\sigma)\overline{f_2(\sigma)}. $$

The k-th moment $m_r(k;n)$ is then expressed as

$$ m_r(k;n) = \langle \mathrm{per}_r(\,\cdot\,;n)^k,\, \mathbf{1}_n\rangle_n. $$

It is thus natural to generalize the situation and study weighted moments. In fact, we can enumerate the character-weighted moments

$$ m_r^\lambda(k;n) := \langle \mathrm{per}_r(\,\cdot\,;n)^k,\, \chi^\lambda\rangle_n $$

for any Young diagram λ.

Our desired enumeration for weighted moments is given as follows:

Theorem 6.1. *For any $n,k,r\in\mathbf{N}$ and any Young diagram λ with n boxes, the character-weighted k-th moment is given by*

$$ m_r^\lambda(k;n) = \sum_{0<rj\le n} \begin{Bmatrix} k \\ j \end{Bmatrix} r^{k-j} R^{(r^j)}_{\lambda(n-rj)}. \tag{6.1} $$

The number $R^\eta_{\lambda\mu}$ ($|\lambda| = |\mu| + |\eta|$) is given by the "Murnaghan-Nakayama rule"

$$ R^\eta_{\lambda\mu} = \sum_S (-1)^{\mathrm{ht}(S)}, $$

where S runs thorough all sequences of diagrams $S = (\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(m)} = \lambda)$ with each $\lambda^{(i)} - \lambda^{(i-1)}$ a border strip of length η_i, $\mathrm{ht}(S)$ is the sum of heights $\mathrm{ht}(\lambda^{(i)}-\lambda^{(i-1)})$ of skew diagrams $\lambda^{(i)}-\lambda^{(i-1)}$ (for the precise meaning, see [4]). In particular, the moment of level 1 is expressed by the Kostka numbers $K_{\lambda\mu}$ as

$$ m^\lambda(k;n) = \sum_{j=1}^{n} \begin{Bmatrix} k \\ j \end{Bmatrix} K_{\lambda(n-j,1^j)}. \tag{6.2} $$

Remark 6.1. This $R^\eta_{\lambda\mu}$ represents also the coefficients of the Schur function $s_\lambda(x)$ in the expansion of the product $p_\eta(x)s_\mu(x)$:

$$ p_\eta(x)s_\mu(x) = \sum_\lambda R^\eta_{\lambda\mu} s_\lambda(x), $$

where $p_\eta(x)$ is the power-sum symmetric function (see [4, p. 48]).

Proof of Theorem 6.1. Consider the set

$$ \mathrm{Conj}_r(\mathfrak{S}_n) = \{1^{p_1}\cdots n^{p_n} \in \mathrm{Conj}(\mathfrak{S}_n) \mid p_r > 0\}. $$

Notice that the **support** of $\mathrm{per}_r(\,\cdot\,;\mathfrak{S}_n,[n])$ is $\mathrm{Conj}_r(\mathfrak{S}_n)$. Using the bijection

$$ \begin{aligned} p_r : \mathrm{Conj}_r(\mathfrak{S}_n) &\ni 1^{p_1}2^{p_2}\cdots r^{p_r}\cdots \\ &\longmapsto 1^{p_1}2^{p_2}\cdots r^{p_r-1}\cdots \in \mathrm{Conj}(\mathfrak{S}_{n-r}), \end{aligned} $$

we define the map $\pi_r : \mathbf{Z}(\mathfrak{S}_n) \to \mathbf{Z}(\mathfrak{S}_{n-r})$ by

$$ (\pi_r\chi)(C) := \chi(p_r^{-1}(C)). $$

In our notation, **the Murnaghan-Nakayama formula** is regarded as the one which describes the irreducible decomposition of $\pi_r\chi^\lambda$:

$$ \pi_r\chi^\lambda = \sum_\mu (-1)^{\mathrm{ht}(\lambda-\mu)}\chi^\mu, \tag{6.3} $$

where μ runs through the diagrams of $n-r$ boxes such that $\lambda-\mu$ is a border strip.

Remark 6.2. The Murnaghan-Nakayama formula can be obtained by calculating the adjoint operator π_r^*.

Let $\psi^{(k)}(\sigma;n,r)$ be the k-th factorial-type product of $\mathrm{per}_r(\,\cdot\,;n)$'s:

$$ \psi^{(k)}(\sigma;n,r) := \prod_{j=0}^{k-1} \left(\mathrm{per}_r(\sigma;n) - rj\right). $$

It is elementary to check the following identity (see e.g. [1]):

$$\mathrm{per}_r(\sigma; n)^k = \sum_{j=1}^{k} \left\{ {k \atop j} \right\} r^{k-j} \psi^{(j)}(\sigma; n, r). \tag{6.4}$$

Consequently, in order to show the theorem, it is enough to calculate the values of the inner products $\langle \psi^{(j)}, \chi^{\lambda} \rangle_n$ in general. In order to achieve this, we deduce a recurrence relation of $\psi^{(j)}$'s as

$$\langle \psi^{(j)}(n, r), \chi^{\lambda} \rangle_n = \langle \psi^{(j-r)}(n-r, r), \pi_r \chi^{\lambda} \rangle_{n-r}.$$

Using the relation above successively, we have

$$\langle \psi^{(j)}(n, r), \chi^{\lambda} \rangle_n = \langle \mathbf{1}_{n-rj}, \pi_r^j \chi^{\lambda} \rangle_{n-r}. \tag{6.5}$$

Applying now the Murnaghan-Nakayama formula (6.3), we get

(6.6)

$$\langle \mathbf{1}_{n-rj}, \pi_r^j \chi^{\lambda} \rangle_{n-rj} = \sum_S (-1)^{\mathrm{ht}(S)} = R^{(r^j)}_{\lambda(n-rj)}.$$

This completes the proof of Theorem 6.1. □

References

[1] Graham, R. L., Knuth, D. E., and Patashnik, O.: Concrete Mathematics. Addison-Wesley, Tokyo (1989).

[2] Kimoto, K.: Selberg zeta functions of infinite symmetric groups. (To appear in Kyushu J. Math.).

[3] Kurokawa, N., and Wakayama, M.: Absolute Casimir Elements. Iwanami, Tokyo (2002). (In Japanese).

[4] MacDonald, I. G.: Symmetric Functions and Hall Polynomials. 2nd ed., Oxford University Press, New York (1995).

[5] Ruelle, D.: Dynamical Zeta Functions for Piecewise Monotone Maps of the Interval. CRM Monograph Series, vol. 4, American Mathematical Society, Providence, RI. (1994).