Fano threefolds with wild conic bundle structures

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Abstract: We classify smooth Fano threefolds with wild conic bundle structures in characteristic 2 without using the general classification methods of Fano threefolds. General results on wild hypersurface bundles of degree p are also obtained.

Key words: Fano threefold; extremal ray; conic bundle.

1. Introduction. In characteristic 0, smooth Fano threefolds with Picard number $\rho = 1$ were classified by V. A. Iskovskikh and V. V. Shokurov. The case $\rho > 1$ was classified by S. Mori and S. Mukai using the classification of $\rho = 1$ case and the extremal rays.

J. Kollár [1] extended the theory of extremal rays [2] to arbitrary positive characteristic. Based on it, N. I. Shepherd-Barron [7] extended the classification of smooth Fano threefolds with $\rho = 1$ to positive characteristic along K. Takeuchi's numerical approach [8].

Once the extremal rays and the classification of $\rho = 1$ case are available, many of the arguments of Mori and Mukai [3–5] for $\rho > 1$ work in positive characteristic because they are basically numerical computations. N. Saito gave a proof of the case $\rho = 2$ in positive characteristic, which was different from the one in [4]. The Mori-Mukai numerical arguments might not work in characteristic 2 for wild conic bundles if the degree of the discriminant locus is involved since the discriminant locus does not make sense and its 'virtual' degree might be negative.

In this paper, we give a simple treatment of such wild conic bundles in a more general setting. We classify smooth Fano threefolds with wild conic bundle structures (Corollary 8) without using general classification theory of Fano threefolds mentioned above. Combined with our result, we think the arguments of [3–5] work in characteristic 2 as well.

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2. Wild hypersurface bundles of degree *p*. We work over an algebraically closed field k of characteristic $p > 0$. Let X, S be smooth irreducible varieties over k with $\dim S = d$, $\dim X = d + r$ and $f : X \to S$ a projective flat morphism with M a relatively very ample divisor such that X is embedded in $\pi : \mathbf{P}_S(E) \to S$, where $E = f_*M$. Let ξ be the tautological line bundle of $\mathbf{P}_S(E)$.

We say that f is a *wild hypersurface bundle of degree* p if every geometric fiber $f^{-1}(s)$ is defined in $\mathbf{P}(E_s)$ by $x^p = 0$ for some non-zero $x \in E_s$.

In this case, there exists a Cartier divisor L on S such that $X \sim p\xi + \pi^*L$, or X is defined by $\varphi \in H^0(S, E^p \otimes L) \subset H^0(S, S^p(E) \otimes L)$ such that $\mathcal{O}_S\varphi$ is a subbundle of $E^p \otimes L$. In the above, $S^p(E)$ is the p-th symmetric product of E and $E^p \subset S^p(E)$ the p-th power of E, or equivalently $E^p = F^*E$, where $F : S \to S$ is the p-th power endomorphism of S.

Theorem 1. *Let $f : X \to S$ be a wild hypersurface bundle of degree p with embedding $X \subset \mathbf{P}_S(E)$ as above. Then φ induces a surjective $\mathcal{O}_S$-homomorphism*

$$\alpha : T_S \twoheadrightarrow E^p \otimes_{\mathcal{O}_S} L/\mathcal{O}_S\varphi,$$

where T_S denotes the tangent bundle of S.

In particular, $d \geq r + 1$ and if $d = r + 1$ then α is an isomorphism.

Proof. Let U be an arbitrary affine open set of S, and $D \in T_S(U)$ an arbitrary k-derivation of $\mathcal{O}_S(U)$. Since $D(x^p) = 0$ for every $x \in E(U)$, D induces a derivation $D_1 : E^p \to E^p$ on U. D also induces a derivation $D_2 : L \to \mathfrak{K}_S$ into the constant sheaf $\mathfrak{K}_S$ of function field of S. As a tensor product, D induces $D_3 : E^p \otimes_{\mathcal{O}_S} L \to E^p \otimes_{\mathcal{O}_S} \mathfrak{K}_S$ by $D_3(a \otimes b) = D_1(a) \otimes b + a \otimes D_2(b)$ on U because $D_3(ar \otimes b) = D_3(a \otimes rb)$ for $r \in \mathcal{O}_S$.

We thus have an element $D_3(\varphi) + \mathfrak{K}_S\varphi \in (E^p \otimes$

$\mathfrak{K}_S/\mathfrak{K}_S\varphi)(U)$ and hence an $\mathcal{O}_S$-homomorphism $\beta : T_S \to E^p \otimes \mathfrak{K}_S/\mathfrak{K}_S\varphi$.

Since $\mathcal{O}_S\varphi$ is a subbundle of $E^p \otimes L$, we have an injection of $\mathcal{O}_S$-modules

$$E^p \otimes L/\mathcal{O}_S\varphi \hookrightarrow E^p \otimes \mathfrak{K}_S/\mathfrak{K}_S\varphi$$

and see that the image of β is contained in $E^p \otimes L/\mathcal{O}_S\varphi$ because $\beta(D) = D_3(\varphi) + \mathfrak{K}_S\varphi$. Hence we have α as required. We note that if $\varphi = \sum_i a_i x_i^p \otimes \ell$ then $\alpha(D) = \sum_i D(a_i)x_i^p \otimes \ell \bmod \mathcal{O}_S\varphi$.

It remains to show that α is surjective. Let $s \in S$ be a closed point, $\partial_1, \dots, \partial_d$ a basis of the stalk $T_{S,s}$ at s and ℓ a generator of $L \otimes \mathcal{O}_{S,s}$. We can write $\varphi = \varphi' \otimes \ell$ with $\varphi' \in E^p \otimes \mathcal{O}_{S,s}$. The Jacobian criterion says that the simultaneous solutions of $\varphi' = \partial_1\varphi' = \cdots = \partial_d\varphi' = 0$ form the singular locus of X over a neighborhood of s. Since X is assumed nonsingular, there exist no solutions. This means that $\varphi' \otimes \ell, \partial_1\varphi' \otimes \ell, \dots, \partial_d\varphi' \otimes \ell$ form a basis of $E^p \otimes L$, which proves that α is surjective at s. The rest is obvious. $\square$

Remark 2. For a proper morphism $f : X \to S$ to be a wild hypersurface bundle of degree p, the existence of M (or the embedding $X \subset \mathbf{P}_S(E)$) is a non-trivial condition. However if $p = r + 1$ then M always exists because $M \sim_f -K_X$. In case $p = 2$ and $r = 1$, a wild hypersurface bundle f of degree 2 is also called a *wild conic bundle*.

3. Characterization of the base. We maintain the same notation and assumptions as in Theorem 1.

Proposition 3. *Assume that $d = r+1$. Then we have*

$$-K_S \sim pc_1(E) + (r+2)L,$$
$$-K_X \sim (r+2-p)\xi + f^*((p-1)c_1(E) + (r+1)L),$$
$$f_*((-K_{X/S})^{r+1}) \equiv -(p-1)(r+2-p)^r(-K_S),$$
$$f_*((-K_X)^{r+1}) \equiv (pr+1)(r+2-p)^r(-K_S),$$

where $\equiv$ means the numerical equivalence and $(-K_{X/S})^{r+1}$, $(-K_X)^{r+1}$ denote powers as cycles.

Proof. We see $-K_S \sim pc_1(E) + (r+2)L$ by Theorem 1. One can reduce the rest to this equivalence relation through simple computations using the equivalence relations $-K_{\mathbf{P}(E)/S} \sim (r+2)\xi - \pi^*c_1(E)$, $X \sim p\xi + f^*L$. $\square$

Comparing the formula in Proposition 3 with the usual conic bundle case $\Delta_f \equiv -f_*K_{X/S}^2 \equiv -f_*K_X^2 - 4K_S$ (Δ_f is the discriminant locus) [4, Proposition 6.2], we see that the wild conic bundle case is easier than the conic bundle case, and we can virtually set $\Delta_f = -K_S$ for the compatibility of formulas and treatments.

The following is much easier than the usual conic bundle case of Fano threefolds.

Corollary 4. *Assume that $d = r + 1 > 1$ and X is a Fano manifold. Then $r + 2 > p$ and $-K_S$ is ample.*

Proof. Since $K_X|_{\text{fiber of } f} \simeq \mathcal{O}(p - r - 2)$, we have $r + 2 > p$. Since $-K_X$ is ample, $-K_S$ is numerically positive by Proposition 3. Again by ampleness of $-K_X$, the cone of curves $NE(X)$ is spanned by a finite number of irreducible curves [2], and so is its image $NE(S)$. Thus $NE(S)$ is closed and $-K_S$ is ample by Kleiman's criterion. $\square$

4. Classification of *X*. Maintaining the notation of Section 2, we will classify wild hypersurface bundles of degree p in a few settings. Our approach is to study the exact sequence:

$$(*) \qquad 0 \to \mathcal{O}_S \to E^p \otimes L \to T_S \to 0.$$

Proposition 5. *Let $f : X \to S$ be a wild hypersurface bundle of degree p such that $d = r + 1$. Assume that S contains a rational curve with normalization $C \simeq \mathbf{P}^1$ and that the restriction $T_X \otimes \mathcal{O}_C$ of T_S to C has a splitting of the form*

$$T_S \otimes \mathcal{O}_C \simeq \mathcal{O}(-1)^{\oplus a_{-1}} \oplus \mathcal{O}^{\oplus a_0} \oplus \mathcal{O}(1)^{\oplus a_1} \oplus \mathcal{O}(2)^{\oplus a_2} \quad (a_0, \dots, a_2 \geq 0).$$

Then we have the following.

1. *Case $(*) \otimes \mathcal{O}_C$ is non-split: $a_0 = 0$, $a_2 = 1$, $E^p \otimes L \otimes \mathcal{O}_C \simeq \mathcal{O}(-1)^{\oplus a_{-1}} \oplus \mathcal{O}(1)^{\oplus (a_1+2)}$. If $a_{-1} > 0$, then $p = 2$ and there is a section $s : C \to X$ over C such that $(K_X \cdot s(C)) \leq 0$,*
2. *Case $(*) \otimes \mathcal{O}_C$ is split: $p = 2$, $a_{-1} = a_1 = 0$ and $E^2 \otimes L \otimes \mathcal{O}_C \simeq \mathcal{O}^{\oplus(a_0+1)} \oplus \mathcal{O}(2)^{\oplus a_2}$.*

Proof. We note first that $a_2 > 0$ by $T_C \simeq \mathcal{O}_C(2) \subset T_S \otimes \mathcal{O}_C$.

We treat the case $(*) \otimes \mathcal{O}_C$ is non-split. Then

$$E^p \otimes L \otimes \mathcal{O}_C \simeq \mathcal{O}(-1)^{\oplus a_{-1}} \oplus \mathcal{O}^{\oplus a_0} \oplus \mathcal{O}(1)^{\oplus (a_1+2)} \oplus \mathcal{O}(2)^{\oplus (a_2-1)}.$$

Since $E \otimes \mathcal{O}_C$ is a direct sum of line bundles, we see $\deg L \equiv 1 \pmod p$. Hence none of $\mathcal{O}(2), \mathcal{O}$ can be a component of $E^p \otimes L \otimes \mathcal{O}_C$ and $a_0 = 0$ and $a_2 = 1$. Assume that $a_{-1} > 0$. Then $p = 2$ by $-1 \equiv 1 \pmod p$. Let $\deg L \otimes \mathcal{O}_C = 1 + 2x$ for some integer x. Then

$$E \otimes \mathcal{O}_C \simeq \mathcal{O}(-1-x)^{\oplus a_{-1}} \oplus \mathcal{O}(-x)^{\oplus (a_1+2)}.$$

Let $s : C \to \mathbf{P}_S(E)$ be the section of π over C such that $s^*\xi$ corresponds to the projection of $E \otimes \mathcal{O}_C$ to $\mathcal{O}(-1-x)$. Then $(s(C) \cdot \xi) = -1-x$ and $(s(C) \cdot c_1(E)) = -(1+x)a_{-1} - x(a_1+2)$. Thus $(s(C) \cdot X) = (s(C) \cdot 2\xi + \pi^* L) = -1 < 0$. Hence $s(C) \subset X$ and we see that $(s(C) \cdot -K_X) = 1 - a_{-1} \leq 0$ by $r = a_{-1} + a_1$ and Proposition 3.

It remains to treat the case $(*) \otimes \mathcal{O}_C$ is split. Then

$$E^p \otimes L \otimes \mathcal{O}_C \simeq \mathcal{O}(-1)^{\oplus a_{-1}} \oplus \mathcal{O}^{\oplus (a_0+1)} \oplus \mathcal{O}(1)^{\oplus a_1} \oplus \mathcal{O}(2)^{\oplus a_2}.$$

We have similarly $a_{-1} = a_1 = 0$ and $p = 2$ by $2 \equiv 0 \pmod p$. □

The following was proved in [7, Proposition 10.1]. We give a different proof.

Corollary 6. *Let X be a Fano threefold in characteristic 2 with a wild conic bundle structure $f : X \to S$. Then S is isomorphic to $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$.*

Proof. By Corollary 4, S is a del Pezzo surface. Hence it is enough to derive a contradiction assuming that there is a (-1)-curve C on S.

Since $T_S \otimes \mathcal{O}_C \simeq \mathcal{O}(-1) \oplus \mathcal{O}(2)$, we have a contradiction by Proposition 5. □

We can classify wild hypersurface bundle of degree p over simple bases.

Theorem 7. *Let $f : X \to S$ be a wild hypersurface bundle of degree p such that $d = r+1 > 1$ and keep the notation of Section 2. If S is a product of projective spaces, we have only two cases.*

1. *Case $S \simeq \mathbf{P}^d$: $E \simeq \mathcal{O}_S^{\oplus (d+1)}$ and $L \simeq \mathcal{O}_S(1)$ modulo tensoring E with a line bundle on S, and X is a smooth divisor of $\mathbf{P}^d \times \mathbf{P}^d$ of bidegree $(1,p)$.*
2. *Case $S \simeq (\mathbf{P}^1)^d$: $p = 2$, $E \simeq \mathcal{O}_S \oplus \left(\bigoplus_{i=1}^d p_i^*\mathcal{O}(1)\right)$ and $L \simeq \mathcal{O}_S$ modulo tensoring E with a line bundle on S, where $p_i : S \to \mathbf{P}^1$ is the i-th projection. X is a divisor of $\mathbf{P}_S(E)$ such that $X \sim 2\xi$.*

Proof. First, we consider the case $S \simeq \mathbf{P}^d$. Let $C \subset S$ be a line. Then $T_S \otimes \mathcal{O}_C \simeq \mathcal{O}(1)^{\oplus (d-1)} \oplus \mathcal{O}(2)$. Then $(*) \otimes \mathcal{O}_C$ is non-split by Proposition 5. Hence $(*)$ is non-split and $(*)$ is the standard exact sequence for T_S and $E^p \otimes L \simeq \mathcal{O}_S(1)^{\oplus (d+1)}$ since $H^1(S, (T_S)^*) \simeq k$. Thus $E^p \simeq \mathcal{O}_S^{\oplus (d+1)}$ and $L \simeq \mathcal{O}_S(1)$ (modulo tensoring E with a line bundle). It remains to show that $E \simeq \mathcal{O}_S^{\oplus (d+1)}$. This follows once we show that E is a direct sum of line bundles. Since $\mathcal{O}_S$ is a direct summand of $F_*\mathcal{O}_S \simeq \oplus_{0 \leq i,j < p} \mathcal{O}_S(-i-j)$, E is a direct summand of $F_*(F^*E) \simeq E \otimes F_*\mathcal{O}_S$. Since $F^*E \simeq F^*$(direct sum of line bundles), F_*F^*E is also a direct sum of line bundles. Hence E is a direct sum of line bundles and we are done in this case.

Next, we assume $S \simeq \mathbf{P}^n \times T$ with $n, \dim T > 0$, and let C be (a line) $\times$ (a point). Then $T_S \otimes \mathcal{O}_C \simeq \mathcal{O}^{\oplus \dim T} \oplus \mathcal{O}(1)^{\oplus (n-1)} \oplus \mathcal{O}(2)$. Hence $(*) \otimes \mathcal{O}_C$ is split, $p = 2$ and $n = 1$ by Proposition 5. Hence S is the product of $\mathbf{P}^1$.

We note $T_S \simeq \bigoplus_{i=1}^d p_i^*\mathcal{O}(2)$ and $H^1(S, (T_S)^*) \simeq \oplus^d p_i^* H^1(\mathbf{P}^1, \mathcal{O}(-2))$. Hence the splitting of the restrictions of $(*)$ above implies the splitting of $(*)$. Hence $E^2 \otimes L \simeq \mathcal{O} \oplus \left(\bigoplus_{i=1}^d p_i^*\mathcal{O}(2)\right)$, and $E^2 \simeq \mathcal{O} \oplus \left(\bigoplus_{i=1}^d p_i^*\mathcal{O}(2)\right)$ and $L \simeq \mathcal{O}_S$. The rest is similar to the case $S \simeq \mathbf{P}^d$. □

Corollary 8. *Let X be a Fano threefold in characteristic 2 with a wild conic bundle structure $f : X \to S$. Then we have one of the following.*

1. *Case $S \simeq \mathbf{P}^2$: X is a divisor of $\mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(1,2)$.*
2. *Case $S \simeq \mathbf{P}^1 \times \mathbf{P}^1$: X is a divisor of $P = \mathbf{P}_S(\mathcal{O}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O})$ such that $X \sim 2\xi$.*

This follows from Theorem 7 except that we need to show that X given in case (2) is Fano. This is however obvious from $-K_X \sim \xi + f^*\mathcal{O}(1,1)$.

Remark 9. The case (1) was proved and the case (2) was mentioned as an example in [6]. The case (1) should 'belong' to [3, Table 2, n° 24], and the case (2) to Table 3, n° 10, which are in characteristic 0. In the case (2), the equivalence of our description and those in [3, 6] requires some explanation.

Remark 10. Let S, P, X be as in Corollary 8, Case (2). These can be described as follows: Let x_0, x_1 (resp. y_0, y_1) be variables of bidegree $(1,0)$ (resp. $(0,1)$). Let Z_0, Z_1, Z_2 be variables of bidegree $(-1,0)$, $(0,-1)$, $(0,0)$. Let $a_0(x_0,x_1)$, $a_1(y_0,y_1)$ be homogeneous polynomials of degree 2 without multiple factors. Then X is defined by

$$a_0(x)Z_0^2 + a_1(y)Z_1^2 + Z_2^2 = 0.$$

For each $i, j = 0, 1$, we consider the open set $U_{i,j} \subset S = \mathbf{P}^1 \times \mathbf{P}^1$ defined by $x_i y_j \neq 0$, and $X_{i,j}$ over $U_{i,j}$ is given by coordinates $(x_{1-i}/x_i, y_{1-j}/y_j) \times (x_i Z_0 : y_j Z_1 : Z_2)$ with one equation

$$a_0\left(\frac{x}{x_i}\right)(x_i Z_0)^2 + a_1\left(\frac{y}{y_j}\right)(y_j Z_1)^2 + Z_2^2 = 0.$$

Furthermore, the birational morphism of X to quadric

$$\{(z_0 : \cdots : z_4) \in \mathbf{P}^4 \mid a_0(z_0, z_1) + a_1(z_2, z_3) + z_4^2 = 0\}$$

in [3, Table 3, n$^{\mathrm{o}}$ 10] is given by $(x_0Z_0 : x_1Z_0 : y_0Z_1 : y_1Z_1 : Z_2)$. To be precise, the morphism on $X_{i,j}$ is given by

$$\left(\frac{x_0}{x_i}x_iZ_0 : \frac{x_1}{x_i}x_iZ_0 : \frac{y_0}{y_j}y_jZ_1 : \frac{y_1}{y_j}y_jZ_1 : Z_2\right).$$

It is easy to see that this is actually a morphism and that X is the blow-up of the quadric as described in Table 3, n$^{\mathrm{o}}$ 10.

5. Further discussions. Contrary to the classification theory of Fano threefolds, the method of our paper can be applied only to a very special class of Fano manifolds. However our method is straightforward and rather precise in the area it covers. For instance, combined with the classification of Fano threefolds, it can cover certain ‘wild’ Fano fivefolds.

Let $f : X \to S$ be a wild hypersurface bundle of degree p such that X is a Fano manifold, $\dim X = 5$ and $\dim S = 3$. Then $p = 2, 3$ and S is a Fano threefold by Corollary 4. If we assume further that $p = 2$, then there is an ample divisor H on S such that $-K_S \sim 2H$ by Proposition 3.

By Proposition 5 and further arguments, we can prove that $S = \mathbf{P}^3$ or $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ in the case $p = 2$. In other words, we get only those offered by Theorem 7. This will be published elsewhere.

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