$L_p(1,\chi)$ mod p

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Abstract: In this paper, we compute $L_p(1,\chi) \bmod p$ when χ is the nontrivial character of a real quadratic field. As a result, we give a sufficient condition for Iwasawa invariants $\mu_p(k)$, $\lambda_p(k)$ to vanish when p splits in a real quadratic field k.

Key words: Greenberg's conjecture; Iwasawa invariants; special value of p-adic L-function.

1. Introduction. Let k be a number field and p a prime number. For a $\mathbf{Z}_p$-extension $k = k_0 \subset k_1 \subset \cdots \subset k_n \subset \cdots \subset k_\infty$ with Galois groups $\mathrm{Gal}(k_n/k) \simeq \mathbf{Z}/p^n\mathbf{Z}$, let A_n be the p-Sylow subgroup of the ideal class group of k_n. Then, by Iwasawa, there exists integers $\mu_p(k)$, $\lambda_p(k)$ and $\nu_p(k)$ such that $|A_n| = p^{\lambda_p(k)n+\mu_p(k)p^n+\nu_p(k)}$ for sufficiently large n. Greenberg's conjecture [2] claims that both $\mu_p(k)$, $\lambda_p(k)$ vanishes for the cyclotomic $\mathbf{Z}_p$-extension of any totally real number field k. Several authors studied Greenberg's conjecture when k is a real quadratic field and p is a small prime. But little is known in the case of large primes. In this paper, we give a sufficient condition for the Iwasawa invariants $\mu_p(k)$, $\lambda_p(k)$ to vanish when k is real quadratic, so that we can determine whether Greenberg's conjecture holds for large primes. The followings are the main results of this paper.

Theorem 1. *Let χ be an even Dirichlet character of conductor Δ, and p be an odd prime relatively prime to Δ. Let $L_p(s,\chi)$ be the p-adic L-function associated with χ. Then*

$$\chi(p)^{-1}L_p(1,\chi) \overset{\mathrm{mod}\,p}{\equiv} \sum_{t=1}^{p-1}\left(1+\frac{1}{2}+\cdots+\frac{1}{t}\right) (\chi(rt)+\chi(rt+1)\cdots+\chi(rt+r-1)),$$

where r and s are integers such that $rp+s\Delta=1$.

Corollary 1. *Let D be a square free integer, $k=\mathbf{Q}(\sqrt{D})$ a real quadratic field, and χ the nontrivial character of k. Let Δ be the discriminant of k and p an odd prime which splits in k. Then*

$$v_p\left(\sum_{t=1}^{p-1}\left(1+\cdots+\frac{1}{t}\right)(\chi(rt)+\cdots+\chi(rt+r-1))\right) = 0 \quad \Rightarrow \mu_p(\mathbf{Q}(\sqrt{D})) = \lambda_p(\mathbf{Q}(\sqrt{D})) = 0,$$

where v_p is the valuation of $\mathbf{C}_p^*$ normalized by $|p|_{v_p} = (1/p)$.

2. Proof of theorems. First we prove Theorem 1.

Proof. Let ζ_Δ be a primitive Δ-th root of unity. By Proposition 2 in [3], we see that

$$L_p(1,\chi) \overset{\mathrm{mod}\,p}{\equiv} \frac{\chi(p)}{\Delta}\sum_{i=1}^{\Delta}\frac{-(1-X^{ri})^p+(1-X^{rip})}{p(1-X^i)}\sum_{j=1}^{\Delta}\chi(j)X^{ij}|_{X=\zeta_\Delta}.$$

So

$$L_p(1,\chi) \equiv \frac{\chi(p)}{\Delta p}(A(\zeta_\Delta)-B(\zeta_\Delta)),$$

where $A(X) = \sum_{i=1}^{\Delta}\frac{1-X^{rip}}{1-X^i}\sum_{j=1}^{\Delta}\chi(j)X^{ij}$, $B(X) = \sum_{i=1}^{\Delta}\frac{(1-X^{ri})^p}{1-X^i}\sum_{j=1}^{\Delta}\chi(j)X^{ij}$. First we compute the value of $A(\zeta_\Delta)$:

$$\begin{aligned} A(X) &= \sum_{i=1}^{\Delta}\frac{1-X^{rip}}{1-X^i}\sum_{j=1}^{\Delta}\chi(j)X^{ij} \\ &= \sum_{i=1}^{\Delta}(1+X^i+\cdots+X^{(rp-1)i})\sum_{j=1}^{\Delta}\chi(j)X^{ij} \\ &= \sum_{i=1}^{\Delta}\sum_{t=0}^{rp-1}\sum_{j=1}^{\Delta}\chi(j)X^{(j+t)i} \\ &= \sum_{t=0}^{rp-1}\sum_{j=1}^{\Delta}\chi(j)\sum_{i=1}^{\Delta}X^{(j+t)i}. \end{aligned}$$

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So,

$$A(\zeta_\Delta) = \sum_{t=0}^{rp-1} \chi(t)\Delta = 0.$$

Next we compute the value of $B(\zeta_\Delta)$:

Let $(1-T)^{p-1} = \sum_{t=0}^{p-1} C_t T^t$. Then

$$\begin{aligned}
&B(X) \\
&= \sum_{i=1}^{\Delta} \frac{(1-X^{ri})^p}{1-X^i} \sum_{j=1}^{\Delta} \chi(j) X^{ij} \\
&= \sum_{i=1}^{\Delta} \frac{(1-X^{ri})}{1-X^i}(1-X^{ri})^{p-1} \sum_{j=1}^{\Delta} \chi(j) X^{ij} \\
&= \sum_{i=1}^{\Delta}(1+\cdots+X^{(r-1)i}) \sum_{t=0}^{p-1} C_t X^{rit} \sum_{j=1}^{\Delta} \chi(j) X^{ij} \\
&= \sum_{i=1}^{\Delta}\sum_{t=0}^{p-1}\sum_{j=1}^{\Delta} \chi(j) X^{(rt+j)i} C_t \\
&\quad + \sum_{i=1}^{\Delta}\sum_{t=0}^{p-1}\sum_{j=1}^{\Delta} \chi(j) X^{(rt+j+1)i} C_t \\
&\quad + \cdots + \sum_{i=1}^{\Delta}\sum_{t=0}^{p-1}\sum_{j=1}^{\Delta} \chi(j) X^{(rt+j+r-1)i} C_t \\
&= \sum_{t=0}^{p-1}\sum_{j=1}^{\Delta} \chi(j) C_t \sum_{i=1}^{\Delta} X^{(rt+j)i} \\
&\quad + \sum_{t=0}^{p-1}\sum_{j=1}^{\Delta} \chi(j) C_t \sum_{i=1}^{\Delta} X^{(rt+j+1)i} \\
&\quad + \cdots + \sum_{t=0}^{p-1}\sum_{j=1}^{\Delta} \chi(j) C_t \sum_{i=1}^{\Delta} X^{(rt+j+r-1)i}.
\end{aligned}$$

So

$$B(\zeta_\Delta) = \Delta \sum_{t=0}^{p-1} C_t(\chi(rt)+\chi(rt+1)+\cdots+\chi(rt+r-1)).$$

Hence

$$\begin{aligned}
L_p(1,\chi) &\equiv \frac{\chi(p)}{\Delta p}(A(\zeta_\Delta) - B(\zeta_\Delta)) \\
&\overset{\text{mod } p}{\equiv} \Big(-\frac{\chi(p)}{p}\sum_{t=0}^{p-1} C_t(\chi(rt)+\chi(rt+1)+\cdots \\
&\qquad\qquad + \chi(rt+r-1))\Big).
\end{aligned}$$

Note that $C_t \equiv 1 \pmod p$. By using $C_0 = 1$ and for $t \geq 1$

$$\begin{aligned}
C_t &= \frac{(p-1)!}{(p-1-t)!t!}(-1)^t \\
&\equiv 1 - \left(1+\frac{1}{2}+\cdots+\frac{1}{t}\right)p \pmod{p^2}
\end{aligned}$$

we conclude that

$$\begin{aligned}
L_p(1,\chi) &\overset{\text{mod } p}{\equiv} -\frac{\chi(p)}{p}\sum_{t=0}^{p-1}(\chi(rt)+\cdots+\chi(rt+r-1)) \\
&+ \chi(p)\sum_{t=1}^{p-1}\Big(1+\cdots+\frac{1}{t}\Big)(\chi(rt)+\cdots+\chi(rt+r-1)) \\
&= \chi(p)\sum_{t=1}^{p-1}\Big(1+\cdots+\frac{1}{t}\Big)(\chi(rt)+\cdots+\chi(rt+r-1)).
\end{aligned}$$

This completes the proof. □

Let D_n be the subgroup of A_n consisting of ideal classes represented by products of prime ideals of k_n lying above p. Taya [4] proved the following theorem using a theorem of Greenberg [2].

Theorem 2. *Let k be a totally real number field and p an odd prime number. Assume that p splits completely in k and also that Lepoldt's conjecture is valid for k and p. Then the following are equivalent.*

(1) $\lambda_p(k) = \mu_p(k) = 0$.

(2) $|D_n| = |A_0|p^{v_p(R_p(k))-[k:\mathbf{Q}]+1}$ *for some* $n \geq 0$.

Here $R_p(k)$ is the p-adic regulator of k. When k is a real abelian number field, we can compute $v_p(R_p(k))$ by the theorem of Colmez [1]:

$$\begin{aligned}
&\lim_{s\to 1}(s-1)\zeta_p(s,k) \\
&\qquad = \frac{2^{[k:\mathbf{Q}]-1}h_k R_p(k)}{\sqrt{d_k}}\prod_{\mathfrak{p}|p}(1-N(\mathfrak{p})^{-1}).
\end{aligned}$$

When p splits completely in k, then it follows from the above formula that

$$\begin{aligned}
&\prod_{1\neq\chi\in \mathrm{Gal}(k/\mathbf{Q})} L_p(1,\chi) \\
&\qquad = \frac{2^{[k:\mathbf{Q}]-1}h_k R_p(k)(1-p^{-1})^{[k:\mathbf{Q}]-1}}{\sqrt{d_k}}.
\end{aligned}$$

Note that $v_p(R_p(k)) \geq [k:\mathbf{Q}]-1$, hence if the left hand side of the above formula is a p adic unit, we see that $p \nmid h_k$. Note also that Leopoldt's conjecture holds for any real abelian extension over $\mathbf{Q}$. Now the proof of Corollary 1 follows directly from Theorem 1, Theorem 2 and discussion above.

Remark 1. Note that $\chi(t) = (\Delta/t)$, and $\chi(t) = (t/D)$ when $D \equiv 1 \bmod 4$. Here $(\frac{*}{-})$ is a

Kronecker symbol. For $p, \Delta < 200$ with $p \equiv 1(\Delta)$,

$$L_p(1,\chi) \equiv \sum_{t=1}^{p-1}\left(1+\frac{1}{2}+\cdots+\frac{1}{t}\right)\chi(t) \not\equiv 0 \pmod p$$

except for $p = 181$, $\Delta = 60$. Hence $\mu_p(\mathbf{Q}(\sqrt{D})) = \lambda_p(\mathbf{Q}(\sqrt{D})) = 0$ except for $\mathbf{Q}(\sqrt{15})$ and $p = 181$ when $p, \Delta < 200$. We do not know whether the Greenberg's conjecture holds for $k = \mathbf{Q}(\sqrt{15})$ and $p = 181$.

3. The case of conductor *p*. In this section we turn our attention to the evaluation of $(L_p(1,\chi) \bmod p)$ when the conductor of a Dirichlet character χ is p. Let ω be the Teichmuller character of conductor p.

Theorem 3. *Let $p \equiv 1 \pmod 4$ be a prime number, $\chi = \omega^{(p-1)/2}$ be the nontrivial character for $\mathbf{Q}(\sqrt{p})$. Then we have*

$$\begin{aligned} L_p(1,\chi) &\equiv 2B_{(p-1)/2} \\ &\equiv \sum_{a=1}^{(p-1)/2}\left(\frac{p}{a}\right)\left(\frac{2}{p}a^{p-1}+a^{p-2}\right) \pmod p. \end{aligned}$$

Here $(\frac{\cdot}{\cdot})$ is a Kronecker symbol and B_n is a n-th Bernoulli number.

Proof. The first equality is already known (see the proof of Theorem 5.37 in Washington [5]). By Corollary 5.15 [5], $B_{1,\omega^{(p-3)/2}} \equiv B_{(p-1)/2}/(p-1)/2 \equiv -2B_{(p-1)/2} \pmod p$. From the decomposition $a = \omega(a)\langle a\rangle$,

$$\begin{aligned} &\frac{1}{p}\sum_{a=1}^{p-1}\left(\frac{p}{a}\right)a^{p-1} \\ &= \frac{1}{p}\sum_{a=1}^{p-1}\chi(a)a^{p-1} = \frac{1}{p}\sum_{a=1}^{p-1}\chi(a)\langle a\rangle^{p-1} \\ &= \frac{1}{p}\sum_{a=1}^{p-1}\chi(a)(\langle a\rangle^{p-1}-1) \\ &= \frac{1}{p}\sum_{a=1}^{p-1}\chi(a)(\langle a\rangle-1)(\langle a\rangle^{p-2}+\cdots+\langle a\rangle+1) \\ &\equiv (p-1)\frac{1}{p}\sum_{a=1}^{p-1}\chi(a)(\langle a\rangle-1) \\ &\equiv -\frac{1}{p}\sum_{a=1}^{p-1}\chi(a)\langle a\rangle = -\frac{1}{p}\sum_{a=1}^{p-1}\omega^{(p-1)/2}(a)\omega^{-1}(a)a \\ &= -B_{1,\omega^{(p-3)/2}} \equiv 2B_{(p-1)/2} \pmod p, \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{p}\sum_{a=1}^{p-1}\left(\frac{p}{a}\right)a^{p-1} \\ &= \frac{1}{p}\sum_{a=1}^{(p-1)/2}\left(\frac{p}{a}\right)a^{p-1} + \frac{1}{p}\sum_{a=1}^{(p-1)/2}\left(\frac{p}{p-a}\right)(p-a)^{p-1} \\ &\equiv \frac{2}{p}\sum_{a=1}^{(p-1)/2}\left(\frac{p}{a}\right)a^{p-1} + \sum_{a=1}^{(p-1)/2}\left(\frac{p}{a}\right)a^{p-2} \pmod p, \end{aligned}$$

which completes the proof. □

Remark 2. Let $p \equiv 1 \pmod 4$ be a prime number, $h, \epsilon = (t+u\sqrt{p})/2 > 1$ be the class number and fundamental unit for $\mathbf{Q}(\sqrt{p})$. Then, by Ankeny-Artin-Chowla,

$$\frac{u}{t}h \equiv B_{(p-1)/2} \pmod p.$$

Since $h < \sqrt{p}$, the above congruence actually determines h if $p \nmid B_{(p-1)/2}$. For $p < 6,270,713$, no examples of $p \nmid B_{(p-1)/2}$ are known (See p. 82 for details in [5]).

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