Generalized isometric spheres and fundamental domains for discrete subgroups of $PU(1,n;\mathbf{C})$

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Abstract: Let G be a discrete subgroup of $PU(1,n;\mathbf{C})$. For a boundary point y of the Siegel domain, we define the generalized isometric sphere $I_y(f)$ of an element f of $PU(1,n;\mathbf{C})$. By using the generalized isometric spheres of elements of G, we construct a fundamental domain $P_y(G)$ for G, which is regarded as a generalization of the Ford domain. And we show that the Dirichlet polyhedron $D(w)$ for G with center w convereges to $P_y(G)$ as $w \to y$. Some results are also found in [5], but our method is elementary.

Key words: Generalized isometric sphere; discrete subgroup; $PU(1,n;\mathbf{C})$.

1. Introduction. Let $\mathbf{C}$ be the field of complex numbers. Let $V = V^{1,n}(\mathbf{C})$ denote the vector space $\mathbf{C}^{n+1}$, together with the unitary structure defined by the Hermitian form

$$\widetilde{\Phi}(z^*, w^*) = -(\overline{z_0^*}w_1^* + \overline{z_1^*}w_0^*) + \sum_{j=2}^{n} \overline{z_j^*}w_j^*$$

for $z^* = (z_0^*, z_1^*, z_2^*, \dots, z_n^*)$, $w^* = (w_0^*, w_1^*, w_2^*, \dots, w_n^*)$ in V. An automorphism g of V, that is a linear bijection such that $\widetilde{\Phi}(g(z^*), g(w^*)) = \widetilde{\Phi}(z^*, w^*)$ for z^*, w^* in V, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1,n;\mathbf{C})$. Set $PU(1,n;\mathbf{C}) = U(1,n;\mathbf{C})/(\text{center})$. Let $V_0 = \{z^* \in V \mid \widetilde{\Phi}(z^*, z^*) = 0\}$ and $V_- = \{z^* \in V \mid \widetilde{\Phi}(z^*, z^*) < 0\}$. It is clear that V_0 and V_- are invariant under $U(1,n;\mathbf{C})$. Set $V^* = V_- \cup V_0 - \{0\}$. Let $\pi : V^* \longrightarrow \pi(V^*)$ be the projection map defined by $\pi(z_0^*, z_1^*, z_2^*, \dots, z_n^*) = (z_1, z_2, \dots, z_n)$, where $z_j = z_j^*/z_0^*$ for $j = 1, 2, \dots, n$. We write ∞ for $\pi(0, 1, 0, \dots, 0)$. We may identify $\pi(V_-)$ with the Siegel domain

$$H^n = \left\{ z = (z_1, z_2, \dots, z_n) \in \mathbf{C}^n \;\middle|\; \mathrm{Re}(z_1) > \frac{1}{2}\sum_{j=2}^{n} |z_j|^2 \right\}.$$

The boundary ∂H^n of the Siegel domain is defined by

$$\partial H^n = \left\{ z = (z_1, z_2, \dots, z_n) \in \mathbf{C}^n \;\middle|\; \mathrm{Re}(z_1) = \frac{1}{2}\sum_{j=2}^{n} |z_j|^2 \right\} \cup \{\infty\}.$$

An element of $PU(1,n;\mathbf{C})$ acts on the Siegel domain H^n and its boundary ∂H^n. Denote $H^n \cup \partial H^n$ by $\overline{H^n}$. In H^n, we can introduce the hyperbolic metric d. An element of $PU(1,n;\mathbf{C})$ is an isometry of H^n with respect to d (see [3, 5] for details).

We concern ourselves with discrete subgroups of $PU(1,n;\mathbf{C})$. In the study of discrete subgroups of $PU(1,n;\mathbf{C})$ it is important to consider their fundamental domains. In this paper we define the generalized isometric spheres of elements of $PU(1,n;\mathbf{C})$, which are used for constructing a fundamental domain for a discrete subgroup of $PU(1,n;\mathbf{C})$. Also we discuss the relationship between this fundamental domain and the Dirichlet polyhedron.

2. Generalized isometric spheres. In this section we give the definition of generalized isometric spheres of elements of $PU(1,n;\mathbf{C})$ and discuss their properties. First we recall some definitions and notation. The *H-coordinates* of a point $(z_1, z_2, \dots, z_n) \in \overline{H^n} - \{\infty\}$ are defined by $(k, t, z')_H \in (\mathbf{R}^+ \cup \{0\}) \times \mathbf{R} \times \mathbf{C}^{n-1}$ such that $k = \mathrm{Re}(z_1) - (1/2)\sum_{j=2}^{n} |z_j|^2$, $t = \mathrm{Im}(z_1)$ and $z' = (z_2, \dots, z_n)$. The *Cygan metric* $\rho(p,q)$ for $p = (k_1, t_1, z')_H$ and $q = (k_2, t_2, Z')_H$ is given by

$$\rho(p,q) = \left| \left\{ \frac{1}{2}\|Z' - z'\|^2 + |k_2 - k_1| \right\} + i\{t_1 - t_2 + \mathrm{Im}(\overline{z'}Z')\} \right|^{\frac{1}{2}},$$

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where $\|Z' - z'\|^2 = \sum_{j=2}^n |Z_j - z_j|^2$ and $\overline{z'}Z' = \sum_{j=2}^n \overline{z_j} Z_j$. We note that this Cygan metric ρ is a generalization of the *Heisenberg metric* δ in ∂H^n (see [7]). The Cygan metric in the usual coordinates can be written as follows:

Proposition 2.1 ([7; Proposition 2.2]). *If* $p = (z_1, z_2, \ldots, z_n)$ *and* $q = (w_1, w_2, \ldots, w_n)$ *in* $\partial H^n - \{\infty\}$, *then*

$$\rho(p,q) = \left|-(\overline{z_1} + w_1) + \sum_{j=2}^n \overline{z_j} w_j\right|^{\frac{1}{2}} = |\widetilde{\Phi}(z^*, w^*)|^{\frac{1}{2}},$$

where $z^* = (1, z_1, z_2, \ldots, z_n) \in \pi^{-1}(p)$ *and* $w^* = (1, w_1, w_2, \ldots, w_n) \in \pi^{-1}(q)$.

Let $f = (a_{ij})_{1 \le i,j \le n+1} \in PU(1,n;\mathbf{C})$ with $f(\infty) \neq \infty$. We use the same symbol f for a unitary transformation, which is a lift of f. The *isometric sphere* $I(f)$ of f is defined by

$$I(f) = \{z = (z_1, z_2, \ldots, z_n) \in \overline{H^n} \mid |\tilde{\Phi}(z^*, q^*)| = |\tilde{\Phi}(z^*, f^{-1}(q^*))|\},$$

where $q^* = (0, 1, 0, \ldots, 0)$, $z^* = (1, z_1, z_2, \ldots, z_n)$ in V^*. This definition does not depend on the choice of a lift of f (see [5, 10, 11]). It follows that the isometric sphere $I(f)$ is the sphere in the Cygan metric with center $f^{-1}(\infty)$ and radius $R_f = \sqrt{1/|a_{12}|}$, that is,

$$I(f) = \left\{ z = (k, t, z')_H \in (\mathbf{R}^+ \cup \{0\}) \times \mathbf{R} \times \mathbf{C}^{n-1} \,\middle|\, \rho(z, f^{-1}(\infty)) = \sqrt{\frac{1}{|a_{12}|}} \right\}.$$

It is easy to see that $R_{f^{-1}} = R_f$. We remark that in the case of $PU(1,1;\mathbf{C})$, our radius of an isometric sphere is the square root of the usual one.

We have the same formulae as in Möbius transformations (see [4]).

Proposition 2.2 ([7; Proposition 2.3]). *Let* g *and* h *be elements with* $g(\infty) \neq \infty$, $h(\infty) \neq \infty$ *and* $gh(\infty) \neq \infty$. *Then*

(1) $R_{gh} = \dfrac{R_g R_h}{\rho(g^{-1}(\infty), h(\infty))}$;

(2) $R_h^2 = \rho((gh)^{-1}(\infty), h^{-1}(\infty))\rho(g^{-1}(\infty), h(\infty))$.

Let y be a point of ∂H^n. For an element f of $PU(1,n;\mathbf{C})$ with $f(y) \neq y$, we define the *generalized isometric sphere* $I_y(f)$ *of* f *at* y as

$$I_y(f) = \{z = (z_1, z_2, \ldots, z_n) \in \overline{H^n} \mid |\tilde{\Phi}(z^*, y^*)| = |\tilde{\Phi}(z^*, f^{-1}(y^*))|\},$$

where $y^* \in \pi^{-1}(y)$ and $z^* \in \pi^{-1}(z)$. We note that if $y = \infty$, then $I_\infty(f)$ is the usual isometric sphere $I(f)$.

Set

$$\alpha_y(f,z) = \frac{|\widetilde{\Phi}(z^*, y^*)|^{\frac{1}{2}}}{|\widetilde{\Phi}(z^*, f^{-1}(y^*))|^{\frac{1}{2}}},$$

for $z^* = (z_0^*, z_1^*, \ldots, z_n^*) \in \pi^{-1}(z)$, $y^* = (y_0^*, y_1^*, \ldots, y_n^*) \in \pi^{-1}(y)$ and $f^{-1}(y^*) = (f^{-1}(y_0^*), f^{-1}(y_1^*), \ldots, f^{-1}(y_n^*)) \in \pi^{-1}(f^{-1}(y))$. By definition, $I_y(f) = \{z \in \overline{H^n} \mid \alpha_y(f,z) = 1\}$. It is easy to see that

$$|y_0^*|^{\frac{1}{2}} |z_0^*|^{\frac{1}{2}} \rho(z,y) = |\widetilde{\Phi}(z^*, y^*)|^{\frac{1}{2}},$$

and

$$|f^{-1}(y_0^*)|^{\frac{1}{2}} |z_0^*|^{\frac{1}{2}} \rho(z, f^{-1}(y)) = |\widetilde{\Phi}(z^*, f^{-1}(y^*))|^{\frac{1}{2}}.$$

In addition we have

$$\frac{R_f}{\rho(y, f(\infty))} = \frac{|\widetilde{\Phi}(y^*, q^*)|^{\frac{1}{2}}}{|\widetilde{\Phi}(f^{-1}(y^*), q^*)|^{\frac{1}{2}}}, = \frac{|y_0^*|^{\frac{1}{2}}}{|f^{-1}(y_0^*)|^{\frac{1}{2}}}$$

where $q^* = (0, 1, 0, \ldots, 0)$. This leads to

$$\alpha_y(f,z) = \frac{|\widetilde{\Phi}(z^*, y^*)|^{\frac{1}{2}}}{|\widetilde{\Phi}(z^*, f^{-1}(y^*))|^{\frac{1}{2}}} = \frac{R_f \rho(z,y)}{\rho(z, f^{-1}(y))\rho(y, f(\infty))}.$$

Thus we have

Proposition 2.3.

$$I_y(f) = \left\{ z \in \overline{H^n} \,\middle|\, \frac{\rho(z,y)}{\rho(z, f^{-1}(y))} = \frac{\rho(y, f(\infty))}{R_f} \right\}.$$

Remark 2.4. In the case of $PU(1,1;\mathbf{C})$, the generalized isometric sphere $I_y(f)$ is an Apollonius circle, because it is the locus of points which have the constant ratio between the Euclidean distances from y and $f^{-1}(y)$.

Let γ be an element of $PU(1,n;\mathbf{C})$ with $\gamma(y) = \infty$. It is easy to show that $z \in \gamma^{-1}(I(\gamma f \gamma^{-1}))$ if and only if $\gamma(z) \in I(\gamma f \gamma^{-1}))$. We see that

$$\frac{|\widetilde{\Phi}(\gamma(z^*), q^*)|^{\frac{1}{2}}}{|\widetilde{\Phi}(\gamma(z^*), \gamma f^{-1}\gamma^{-1}(q^*)|^{\frac{1}{2}}} = \frac{|\widetilde{\Phi}(\gamma(z^*), \gamma(y^*))|^{\frac{1}{2}}}{|\widetilde{\Phi}(\gamma(z^*), \gamma f^{-1}(y^*)|^{\frac{1}{2}}} = \frac{|\widetilde{\Phi}(z^*, y^*)|^{\frac{1}{2}}}{|\widetilde{\Phi}(z^*, f^{-1}(y^*))|^{\frac{1}{2}}}.$$

Thus we have

Proposition 2.5 (cf. [1]). *For any element* γ *of* $PU(1,n;\mathbf{C})$ *with* $\gamma(y) = \infty$,

$$
\begin{aligned}
I_y(f) &= \gamma^{-1}(I(\gamma f\gamma^{-1})) \\
&= \{ z \in \overline{H^n} \mid \rho(\gamma(z), \gamma f^{-1}\gamma^{-1}(\infty)) = R_{\gamma f\gamma^{-1}} \}.
\end{aligned}
$$

We shall show the basic properties of $\alpha_y(f,z)$.

Proposition 2.6 (cf. [1, p. 66]). *Let f and g be elements of $PU(1,n;\mathbf{C})$. Then*

(1) $\alpha_\infty(f,z) = \frac{R_f}{\rho(z, f^{-1}(\infty))}$;

(2) $\alpha_y(f,z) = \alpha_{g(y)}(gfg^{-1}, g(z))$;

(3) $\alpha_y(fg,z) = \alpha_y(f, g(z))\alpha_y(g,z)$;

(4) $\alpha_y(f,z) = \frac{\rho(z,y)}{\rho(f(z),y)}\alpha_\infty(f,z)$;

(5) $\alpha_y(f,z) = \alpha_z(f^{-1}, y)$.

Proof. (1) This is immediate.

(2) There is an element γ such that $\gamma^{-1}(\infty) = g(y)$. By definition,

$$
\begin{aligned}
\alpha_{g(y)}(gfg^{-1}, g(z)) &= \frac{|\widetilde{\Phi}(g(z^*), g(y^*))|^{\frac12}}{|\widetilde{\Phi}(g(z^*), gf^{-1}g^{-1}(g(y^*)))|^{\frac12}} \\
&= \frac{|\widetilde{\Phi}(z^*, y^*)|^{\frac12}}{|\widetilde{\Phi}(z^*, f^{-1}(y^*))|^{\frac12}} \\
&= \alpha_y(f,z).
\end{aligned}
$$

(3) Similarly, we have

$$
\begin{aligned}
\alpha_y(fg,z) &= \frac{|\widetilde{\Phi}(z^*, y^*)|^{\frac12}}{|\widetilde{\Phi}(z^*, g^{-1}f^{-1}(y^*))|^{\frac12}} \\
&= \frac{|\widetilde{\Phi}(z^*, y^*)|^{\frac12}|\widetilde{\Phi}(g(z^*), y^*)|^{\frac12}}{|\widetilde{\Phi}(z^*, g^{-1}(y^*))|^{\frac12}|\widetilde{\Phi}(g(z^*), f^{-1}(y^*))|^{\frac12}} \\
&= \alpha_y(g,z)\alpha_y(f,g(z)).
\end{aligned}
$$

(4) We have

$$
\begin{aligned}
&\alpha_\infty(f,z)\frac{\rho(z,y)}{\rho(f(z),y)} \\
&= \frac{|\widetilde{\Phi}(z^*, q^*)|^{\frac12}}{|\widetilde{\Phi}(z^*, f^{-1}(q^*))|^{\frac12}} \left(\frac{|\widetilde{\Phi}(z^*, y^*)|^{\frac12}}{|z_0^*|^{\frac12}|y_0^*|^{\frac12}} \right) \\
&\qquad\qquad \times \left(\frac{|f(z^*)_0|^{\frac12}|y_0^*|^{\frac12}}{|\widetilde{\Phi}(f(z^*), y^*)|^{\frac12}} \right) \\
&= \frac{|\widetilde{\Phi}(z^*, q^*)|^{\frac12}}{|\widetilde{\Phi}(z^*, f^{-1}(q^*))|^{\frac12}} \left(\frac{|\widetilde{\Phi}(z^*, y^*)|^{\frac12}|\widetilde{\Phi}(f(z^*), q^*)|^{\frac12}}{|\widetilde{\Phi}(f(z^*), y^*)|^{\frac12}|\widetilde{\Phi}(z^*, q^*)|^{\frac12}} \right) \\
&= \frac{|\widetilde{\Phi}(z^*, y^*)|^{\frac12}}{|\widetilde{\Phi}(z^*, f^{-1}(y^*))|^{\frac12}} = \alpha_y(f,z).
\end{aligned}
$$

(5) Likewise, we have

$$
\begin{aligned}
\alpha_y(f,z) = \frac{|\widetilde{\Phi}(z^*, y^*)|^{\frac12}}{|\widetilde{\Phi}(z^*, f^{-1}(y^*))|^{\frac12}} &= \frac{|\widetilde{\Phi}(z^*, y^*)|^{\frac12}}{|\widetilde{\Phi}(f(z^*), y^*)|^{\frac12}} \\
&= \alpha_z(f^{-1}, y).
\end{aligned}
$$

□

Put

$$
\begin{aligned}
\operatorname{Ext} I_y(f) &= \{ z \in \overline{H^n} \mid \alpha_y(f,z) < 1 \}, \\
\operatorname{Int} I_y(f) &= \{ z \in \overline{H^n} \mid \alpha_y(f,z) > 1 \},
\end{aligned}
$$

respectively. The following facts are easily verified:

(1) $y \in \operatorname{Ext} I_y(f)$;

(2) $f(y) \in \operatorname{Int} I_y(f^{-1})$;

(3) $f^{-1}(y) \in \operatorname{Int} I_y(f)$.

Suppose that $y \in \operatorname{Ext} I(f)$. That is,

$$
1 > \alpha_\infty(f,y) = \frac{|\widetilde{\Phi}(y^*, q^*)|^{\frac12}}{|\widetilde{\Phi}(y^*, f^{-1}(q^*))|^{\frac12}} = \frac{1}{\alpha_y(f, f^{-1}(\infty))}.
$$

This is true if and only if $f^{-1} \in \operatorname{Int} I_y(f)$. Therefore we have

Proposition 2.7. *The following* (1) *and* (2) *are equivalent.*

(1) $y \in \operatorname{Ext} I(f)$;

(2) $f^{-1}(\infty) \in \operatorname{Int} I_y(f)$.

It follows from (2) in Proposition 2.6 that $\alpha_{f(y)}(f,z) = \alpha_y(f, f^{-1}(z))$. Hence just as in the case of isometric spheres, we have

Proposition 2.8.

(1) $I_{f(y)}(f) = f(I_y(f)) = I_y(f^{-1})$;

(2) $f(\operatorname{Ext} I_y(f)) \subset \operatorname{Int} I_y(f^{-1})$;

(3) $f(\operatorname{Int} I_y(f)) \subset \operatorname{Ext} I_y(f^{-1})$.

Next we consider the location of fixed points of elements.

Proposition 2.9. *Let f be an element of $PU(1,n;\mathbf{C})$ which does not fix ∞. Let x be a fixed point of f. If f is elliptic or parabolic, then x lies on both $I(f)$ and $I(f^{-1})$. If f is loxodromic, then neither $I(f)$ nor $I(f^{-1})$ contains x.*

Proof. First we consider the case that f is an elliptic element with only one fixed point x in H^n. We may assume that $x = (1,0,\ldots,0)$. Let f be of the form $(a_{ij})_{1\le i,j\le n+1}$. Then we have $I(f^{-1}) = \{ z = (z_1, z_2, \ldots, z_n) \in \overline{H^n} \mid |\overline{a_{22}} + \overline{a_{12}}z_1 - \sum_{j=3}^{n+1}\overline{a_{j2}}z_{j-1}| = 1 \}$. Since f and f^{-1} fix $(1,0,\ldots,0)$,

$$a_{11} + a_{12} = a_{21} + a_{22}; \tag{2.1}$$

$$a_{22} + a_{12} = a_{21} + a_{11}; \tag{2.2}$$

$$a_{k1} + a_{k2} = 0 \quad (k \ge 3). \tag{2.3}$$

It follows from (2.1) and (2.2) that

$$a_{11} = a_{22} \quad \text{and} \quad a_{12} = a_{21}. \tag{2.4}$$

We deduce from (2.3), (2.4) and equations (11), (13) in [6, p. 30] that

$$-2\,\mathrm{Re}(\overline{a_{11}}a_{12}) + \sum_{k=3}^{n+1} |a_{k1}|^2 = 0;$$
$$-|a_{11}|^2 - |a_{12}|^2 - \sum_{k=3}^{n+1} |a_{k1}|^2 = -1.$$

Therefore $|a_{11} + a_{12}| = 1$, which implies $x = (1, 0, \dots, 0) \in I(f^{-1})$. If an elliptic element has more than one fixed point in H^n, then it has a fixed point in ∂H^n. Therefore we have only to treat the case that f has a fixed piont in ∂H^n. Without loss of generality, we may assume that f has a fixed point $\mathbf{0} = (0, 0, \dots, 0) \in \partial H^n$. Then f is of the form

$$f = \begin{pmatrix} \lambda & s & -b \\ 0 & \mu & 0 \\ 0 & -a & A \end{pmatrix},$$

where a, b and A are $(n-1)\times 1$, $1 \times (n-1)$, and $(n-1)\times(n-1)$ matrices, respectively. Furthermore, $\overline{\mu}\lambda = 1$, $\mathrm{Re}(\overline{\mu}s) = (1/2)\|a\|^2$ and $b = \lambda\overline{a}^T A$. Let $a = (a_1, a_2, \dots, a_{n-1})^T$. We have $\rho(f(\infty), 0) = |\mu s^{-1}|^{(1/2)}$ and $R_f = |s|^{-(1/2)}$. Hence $\mathbf{0}$ belongs to $I(f^{-1})$ if and only if $|\mu| = 1$, which means that f is elliptic or parabolic. Let f be elliptic and let $x = (x_1, \dots, x_n)$ be its non-zero fixed point. As f^{-1} fixes x,

$$\frac{\overline{\lambda}x_1}{\overline{\lambda} + \overline{s}x_1 + \sum_{j=1}^{n-1} \overline{a_j}x_{j+1}} = x_1,$$

which yields $\overline{s}x_1 + \sum_{j=1}^{n-1} \overline{a_j}x_{j+1} = 0$. Since $I(f^{-1}) = \bigl\{z = (z_1, z_2, \dots, z_n) \in \overline{H^n} \mid |\overline{\lambda} + \overline{s}z_1 + \sum_{j=1}^{n-1} \overline{a_j}z_{j+1}| = 1\bigr\}$, the fixed point x of f is contained in the isometric sphere $I(f^{-1})$. By the same argument above, we see that x lies on $I(f)$. Thus we have proved the proposition. □

We show that replacing isometric spheres by generalized isometric spheres leads to the same conclusion as in Proposition 2.9.

Proposition 2.10. *Let f be an element of $PU(1, n; \mathbf{C})$ which does not fix either y or ∞. Let x be a fixed point of f. If f is elliptic or parabolic, then x lies on both $I_y(f)$ and $I_y(f^{-1})$. If f is loxodromic, then neither $I_y(f)$ nor $I_y(f^{-1})$ contains x.*

Proof. In a manner similar to Proposition 2.4 of [7], we have

$$\rho(x, f^{-1}(y)) = \rho(f^{-1}(x), f^{-1}(y))$$
$$= \frac{R_f^2\rho(x, y)}{\rho(f(\infty), x)\rho(f(\infty), y)}.$$

It follows that

$$\frac{R_f\rho(x, y)}{\rho(x, f^{-1}(y))\rho(f(\infty), y)} = \frac{\rho(f(\infty), x)}{R_f}.$$

By using Proposition 2.9, we complete our proof. □

3. Fundamental domains. Let G be a discrete subgroup of $PU(1, n; \mathbf{C})$. We define the limit set $L(G)$ of G as the set of points at which one orbit accumulates. The ordinary set $\Omega(G)$ of G is defined as the complement of $L(G)$ in $\overline{H^n}$. Assume that $\infty \in \Omega(G)$ and its stability subgroup $G_\infty = \{\text{identity}\}$. Then there is a positive constant M such that $\rho(0, g(\infty)) \le M$ for any element g of G. Since we have Proposition 2.2 as in the case of Möbius transformations, the same argument as in [4] leads to the following results.

(1) *The radii of isometric spheres are bounded above.*

(2) *The number of isometric spheres with radii exceeding a given positive quantity is finite.*

(3) *Given any infinite sequence of distinct isometric spheres $I(g_1), I(g_2), \dots,$ of elements of G, the radii being R_{g_1}, $R_{g_2}, \dots,$ then* $\lim_{m\to\infty} R_{g_m} = 0$.

By using generalized isometric spheres, we can construct a fundamental domain for a discrete subgroup of $PU(1, n; \mathbf{C})$ as in the Ford domain (see [1, 4, 8, 9]).

Theorem 3.1. *Let G be a discrete subgroup of $PU(1, n; \mathbf{C})$. Let ∞ be a point of $\Omega(G)$ and let $G_\infty = \{\text{identity}\}$. If y is a point of $\Omega(G) \cap \partial H^n$ such that G_y consists only of the identity, then*

$$P_y(G) = \bigcap_{f \in G - \{\mathrm{id}\}} \mathrm{Ext}\, I_y(f)$$

is a fundamental domain for G.

We call $P_y(G)$ the *generalized Ford domain* for G. Let z_1, z_2 be two different points in H^n. Let $E(z_1, z_2)$ be the bisector of $\{z_1, z_2\}$, that is,

$$E(z_1, z_2) = \{w \in H^n \mid d(z_1, w) = d(z_2, w)\},$$

(see [5] for details). Let w be any point of H^n that is not fixed by any element of G except the identity. The Dirichlet polyhedron $D(w)$ for G with center w is defined by

$$D(w) = \bigcap_{g \in G - \{\mathrm{id}\}} H_g(w),$$

where $H_g(w) = \{z \in H^n \mid d(z,w) < d(z,g(w))\}$. We see that

(1) *$D(w)$ is not necessarily convex.*

(2) *$D(w)$ is star-shaped about w.*

(3) *$D(w)$ is locally finite.*

Details and references for these will be found in [2, 12].

We observe the relationship between the Dirichlet polyhedron $D(w)$ and the generalized Ford domain $P_y(G)$.

Theorem 3.2. *Let G be a discrete subgroup of $PU(1,n;\mathbf{C})$. Let $z \in H^n$ and let $y \in \partial H^n \cap \Omega(G)$. Then $D(z) \to P_y(G)$ as $z \to y$.*

To prove Theorem 3.2, we have only to show the following lemma.

Lemma 3.3. *Let f be an element of $PU(1,n;\mathbf{C})$ with $f(y) \neq y$ and $f(\infty) \neq \infty$. Then $E(z, f^{-1}(z))$ converges to $I_y(f)$ as $z \to y$.*

Proof. We have

$$E(z,f^{-1}(z)) = \left\{ w \in H^n \;\middle|\; \frac{|\widetilde{\Phi}(z^*,w^*)|}{|\widetilde{\Phi}(z^*,z^*)|^{\frac{1}{2}}} = \frac{|\widetilde{\Phi}(f^{-1}(z^*),w^*)|}{|\widetilde{\Phi}(f^{-1}(z^*),f^{-1}(z^*))|^{\frac{1}{2}}} \right\}$$

$$= \left\{ w \in H^n \;\middle|\; \frac{|\widetilde{\Phi}(z^*,w^*)|}{|\widetilde{\Phi}(f^{-1}(z^*),w^*)|} = \frac{|\widetilde{\Phi}(z^*,z^*)|^{\frac{1}{2}}}{|\widetilde{\Phi}(f^{-1}(z^*),f^{-1}(z^*))|^{\frac{1}{2}}} \right\},$$

where $z^* \in \pi^{-1}(z)$, $w^* \in \pi^{-1}(w)$ and $f^{-1}(z^*) \in \pi^{-1}(f^{-1}(z))$. We see that

$$\frac{|\widetilde{\Phi}(z^*,w^*)|}{|\widetilde{\Phi}(f^{-1}(z^*),w^*)|} \to \frac{|\widetilde{\Phi}(y^*,w^*)|}{|\widetilde{\Phi}(f^{-1}(y^*),w^*)|}$$

as $z \to y$. Thus $E(z,f^{-1}(z))$ converges to $I_y(f)$ as $z \to y$. □

From the manner of constructing $P_y(G)$, we have

Corollary 3.4. *The fundamental domain $P_y(G)$ is locally finite.*

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