Non-immersion theorems for warped products in complex hyperbolic spaces

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Abstract: We prove a general optimal inequality for warped products in complex hyperbolic spaces and investigate warped products which satisfy the equality case of the inequality. As immediate applications, we obtain several non-immersion theorems for warped products in complex hyperbolic spaces.

Key words: Warped products; inequality; complex hyperbolic space; non-immersion theorem; minimal immersion.

1. Introduction. Let N_1 and N_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and let f be a positive differentiable function on N_1. The warped product $N_1 \times_f N_2$ is defined to be the product manifold $N_1 \times N_2$ equipped with the Riemannian metric given by $g_1 + f^2 g_2$ (see [6]).

For a warped product $N_1 \times_f N_2$, we denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the distributions given by the vectors tangent to leaves and fibers, respectively. Thus, $\mathcal{D}_1$ is obtained from tangent vectors of N_1 via the horizontal lift and $\mathcal{D}_2$ from tangent vectors of N_2 via the vertical lift.

Let $\phi : N_1 \times_f N_2 \to CH^m(4c)$ be an isometric immersion of a warped product $N_1 \times_f N_2$ into a complex hyperbolic m-space with constant holomorphic sectional curvature $4c, c < 0$. Denote by h the second fundamental form of ϕ. Let $\operatorname{trace} h_1$ and $\operatorname{trace} h_2$ be the trace of h restricted to N_1 and N_2, i.e.,

$$\operatorname{trace} h_1 = \sum_{\alpha=1}^{n_1} h(e_\alpha, e_\alpha),$$

$$\operatorname{trace} h_2 = \sum_{t=n_1+1}^{n_1+n_2} h(e_t, e_t)$$

for some orthonormal frame fields $e_1, \dots, e_{n_1}$ and $e_{n_1+1}, \dots, e_{n_1+n_2}$ of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. The immersion ϕ is called *mixed totally geodesic* if

$$h(X, Z) = 0$$

for any X in $\mathcal{D}_1$ and Z in $\mathcal{D}_2$.

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In this article we prove the following general result for arbitrary isometric immersions of warped products into complex hyperbolic spaces.

Theorem 1. *Let $\phi : N_1 \times_f N_2 \to CH^m(4c)$ be an arbitrary isometric immersion of a warped product $N_1 \times_f N_2$ into the complex hyperbolic m-space $CH^m(4c)$. Then we have*

$$\frac{\Delta f}{f} \leq \frac{(n_1+n_2)^2}{4n_2} H^2 + n_1 c, \tag{1.1}$$

where $n_i = \dim N_i$, $i = 1, 2$, H^2 is the squared mean curvature of ϕ, and Δ is the Laplacian operator of N_1.

The equality sign of (1.1) *holds if and only if the following three conditions are satisfied*:

(1) *ϕ is mixed totally geodesic,*

(2) $\operatorname{trace} h_1 = \operatorname{trace} h_2$, *and*

(3) *$J\mathcal{D}_1 \perp \mathcal{D}_2$, where J is the almost complex structure of CH^m.*

As interesting applications of Theorem 1 we have the following non-immersion theorems.

Theorem 2. *Let $N_1 \times_f N_2$ be a warped product whose warping function f is a harmonic function. Then $N_1 \times_f N_2$ does not admit an isometric minimal immersion into any complex hyperbolic space.*

Theorem 3. *If f is an eigenfunction of the Laplacian on N_1 with eigenvalue $\lambda > 0$, then $N_1 \times_f N_2$ does not admits an isometric minimal immersion into any complex hyperbolic space.*

Theorem 4. *If N_1 is a compact Riemannian manifold, then every warped product $N_1 \times_f N_2$ does not admit an isometric minimal immersion into any complex hyperbolic space.*

Theorem 2 is a generalization of a result of N. Ejiri [5]. Also Theorems 2, 3 and 4 can be regarded as partial extensions of Theorems 2, 3 and 4 of [3].

2. Preliminaries. A Kaehler manifold $\tilde{M}^m(4c)$ of constant holomorphic sectional curvature $4c$ is called a *complex space form*. Let N be an n-dimensional Riemannian manifold isometrically immersed in $\tilde{M}^m(4c)$ with $n \geq 2$. We denote by $\langle\ ,\ \rangle$ the inner product for N as well as for $\tilde{M}^m(4c)$.

For any vector X tangent to N we put

$$JX = PX + FX,$$

where PX and FX are the tangential and the normal components of JX, respectively. Thus, P is a well-defined endomorphism of the tangent bundle TN.

We denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of N and $\tilde{M}^m(4c)$, respectively. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for vector fields X, Y tangent to N and ξ normal to N, where h denotes the second fundamental form, D the normal connection, and A the shape operator of the submanifold.

The mean curvature vector $\overrightarrow{H}$ is defined by

$$\overrightarrow{H} = \frac{1}{n}\,\mathrm{trace}\, h.$$

The squared mean curvature is given by

$$H^2 = \langle \overrightarrow{H}, \overrightarrow{H} \rangle.$$

The submanifold N is called *minimal* if its mean curvature vector vanishes identically.

Denote by $K(e_i \wedge e_j)$ the sectional curvature of the plane section spanned by $e_i, e_j; 1 \leq i < j \leq n$. The scalar curvature of N is then given by

$$\tau = \sum_{i<j} K(e_i \wedge e_j).$$

For a differentiable function φ on N, the Laplacian of φ is defined by

$$\Delta\varphi = \sum_{j=1}^{n} \{(\nabla_{e_j} e_j)\varphi - e_j e_j \varphi\},$$

where $e_1, \ldots, e_n$ is an orthonormal frame. When N is compact, each eigenvalue of Δ is non-negative.

3. Proofs of Theorems. Let $\phi : N_1 \times_f N_2 \to CH^m(4c)$ be an isometric immersion of a warped product $N_1 \times_f N_2$ into the complex hyperbolic m-space $CH^m(4c)$. Denote by n_1, n_2 and n the dimensions of N_1, N_2 and $N_1 \times N_2$, respectively. We use the following convention on the range of indices unless mentioned otherwise:

$$j, k, \ell = 1, \ldots, n;$$
$$\alpha, \beta = 1, \ldots, n_1;$$
$$s, t = n_1 + 1, \ldots, n_1 + n_2.$$

Since $N_1 \times_f N_2$ is a warped product, we have [2, 6]

$$\nabla_X Z = \nabla_Z X = (X \ln f) Z, \quad \langle \nabla_X Y, Z \rangle = 0 \tag{3.1}$$

for unit vector fields X, Y in $\mathcal{D}_1$ and Z in $\mathcal{D}_2$. Hence, from (3.1), we find

$$\begin{aligned} K(X \wedge Z) &= \langle \nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z \rangle \\ &= \frac{1}{f}\{(\nabla_X X) f - X^2 f\}. \end{aligned} \tag{3.2}$$

If we choose a local field of orthonormal frame $e_1, \ldots, e_{n_1+n_2}$ such that $e_1, \ldots, e_{n_1}$ are in $\mathcal{D}_1$ and $e_{n_1+1}, \ldots, e_{n_1+n_2}$ in $\mathcal{D}_2$, then (3.2) implies that

$$\frac{\Delta f}{f} = \sum_{\alpha=1}^{n_1} K(e_\alpha \wedge e_s), \quad s = n_1 + 1, \ldots, n_1 + n_2. \tag{3.3}$$

Let R denote the Riemannian curvature tensor of N. The equation of Gauss is given by

$$\begin{aligned} &\langle R(X,Y)Z, W \rangle \\ &= \langle h(X,W), h(Y,Z) \rangle - \langle h(X,Z), h(X,Z) \rangle \\ &\quad + c\{\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle \\ &\quad + \langle JY, Z \rangle \langle JX, W \rangle - \langle JX, Z \rangle \langle JY, W \rangle \\ &\quad + 2 \langle X, JY \rangle \langle JZ, W \rangle\}. \end{aligned} \tag{3.4}$$

It follows from (3.4) that the scalar curvature and the squared mean curvature of N satisfy

$$2\tau = n^2 H^2 - ||h||^2 + n(n-1)c + 3c\,||P||^2, \tag{3.5}$$

where $n = n_1 + n_2$ and $||h||^2$ denotes the squared norm of the second fundamental form and

$$||P||^2 = \sum_{i,j=1}^{n} \langle e_i, P e_j \rangle^2$$

is the squared norm of the endomorphism P.

If we put

$$\eta = 2\tau - \frac{n^2}{2}H^2 - n(n-1)c - 3c||P||^2, \tag{3.6}$$

then we obtain from (3.5) and (3.6) that

$$n^2H^2 = 2\eta + 2||h||^2. \tag{3.7}$$

If we choose a local field of orthonormal frame $e_{n+1}, \dots, e_{2m}$ of the normal bundle so that e_{n+1} is in the direction of the mean curvature vector, then (3.7) becomes

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\Big\{\eta + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\Big\}. \tag{3.8}$$

(3.8) can be restated as

$$\begin{aligned}(a_1+a_2+a_3)^2 &= 2\Big\{\eta + a_1^2 + a_2^2 + a_3^2 + 2\sum_{1\le i<j\le n}(h_{ij}^{n+1})^2 \\ &+ \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 - 2\sum_{2\le\alpha<\beta\le n_1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} \\ &- 2\sum_{n_1+1\le s<t\le n} h_{ss}^{n+1}h_{tt}^{n+1}\Big\},\end{aligned} \tag{3.9}$$

where

$$\begin{aligned} a_1 &= h_{11}^{n+1}, \\ a_2 &= h_{22}^{n+1} + \cdots + h_{n_1n_1}^{n+1}, \\ a_3 &= h_{n_1+1n_1+1}^{n+1} + \cdots + h_{nn}^{n+1}. \end{aligned} \tag{3.10}$$

Thus, by applying Lemma 1 of [1] to (3.9) we find

$$\begin{aligned} &2h_{11}^{n+1}(h_{22}^{n+1} + \cdots + h_{n_1n_1}^{n+1}) \\ &\ge \eta + 2\sum_{1\le i<j\le n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 \\ &- 2\sum_{2\le\alpha<\beta\le n_1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} - 2\sum_{n_1+1\le s<t\le n} h_{ss}^{n+1}h_{tt}^{n+1}\end{aligned} \tag{3.11}$$

which is nothing but

$$\begin{aligned} &\sum_{1\le\alpha<\beta\le n_1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} + \sum_{n_1+1\le s<t\le n} h_{ss}^{n+1}h_{tt}^{n+1} \\ &\ge \frac{\eta}{2} + \sum_{1\le j<k\le n}(h_{jk}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m}\sum_{j,k=1}^{n}(h_{jk}^{r})^2. \end{aligned} \tag{3.12}$$

The equality sign of (3.12) holds if and only if the following condition:

$$h_{11}^{n+1} + \cdots + h_{n_1n_1}^{n+1} = h_{n_1+1n_1+1}^{n+1} + \cdots + h_{nn}^{n+1} \tag{3.13}$$

holds.

By applying (3.3) and (3.4) of Gauss, we have

$$\begin{aligned} \frac{n_2\Delta f}{f} &= \tau - \sum_{\alpha<\beta} K(e_\alpha\wedge e_\beta) - \sum_{s<t} K(e_s\wedge e_t) \\ &= \tau - \frac{n_1(n_1-1)}{2}c - \frac{n_2(n_2-1)}{2}c \\ &- \sum_{r=n+1}^{2m}\sum_{\alpha<\beta}\big(h_{\alpha\alpha}^{r}h_{\beta\beta}^{r} - (h_{\alpha\beta}^{r})^2\big) \\ &- \sum_{r=n+1}^{2m}\sum_{s<t}\big(h_{ss}^{r}h_{tt}^{r} - (h_{st}^{r})^2\big) \\ &- \sum_{\alpha<\beta} 3c\langle Pe_\alpha, e_\beta\rangle^2 - \sum_{s<t} 3c\langle Pe_s, e_t\rangle^2. \end{aligned} \tag{3.14}$$

Therefore, by applying (3.6), (3.12) and (3.14), we obtain

$$\begin{aligned} \frac{n_2\Delta f}{f} &\le \tau - \frac{n(n-1)}{2}c + n_1n_2c - \frac{\eta}{2} \\ &- \sum_{\alpha,t}(h_{\alpha t}^{n+1})^2 - \frac{1}{2}\sum_{r=n+2}^{m}\sum_{j,k=1}^{n}(h_{jk}^{r})^2 \\ &+ \sum_{r=n+2}^{2m}\sum_{\alpha<\beta}\big((h_{\alpha\beta}^{r})^2 - h_{\alpha\alpha}^{r}h_{\beta\beta}^{r}\big) \\ &+ \sum_{r=n+2}^{2m}\sum_{s<t}\big((h_{st}^{r})^2 - h_{ss}^{r}h_{tt}^{r}\big) \\ &- \sum_{\alpha<\beta} 3c\langle Pe_\alpha, e_\beta\rangle^2 - \sum_{s<t} 3c\langle Pe_s, e_t\rangle^2 \\ &= \tau - \frac{n(n-1)}{2}c + n_1n_2c - \frac{\eta}{2} \\ &- \sum_{r=n+1}^{2m}\sum_{\alpha,t}(h_{\alpha t}^{r})^2 - \frac{1}{2}\sum_{r=n+2}^{2m}\Big(\sum_{\alpha} h_{\alpha\alpha}^{r}\Big)^2 \\ &- \frac{1}{2}\sum_{r=n+2}^{2m}\Big(\sum_{t} h_{tt}^{r}\Big)^2 - \sum_{\alpha<\beta} 3c\langle Pe_\alpha, e_\beta\rangle^2 \\ &- \sum_{s<t} 3c\langle Pe_s, e_t\rangle^2, \end{aligned} \tag{3.15}$$

where the equality case of the inequality holds if and only if (3.13) is satisfied.

From (3.15) we find

(3.16)
$$\frac{n_2\Delta f}{f} \le \tau - \frac{n(n-1)}{2}c + n_1 n_2 c - \frac{\eta}{2} - \sum_{\alpha<\beta} 3c\,\langle Pe_\alpha, e_\beta\rangle^2 - \sum_{s<t} 3c\,\langle Pe_s, e_t\rangle^2$$

with the equality holding if and only we have

$$h(TN_1, TN_2) = \{0\}, \quad \sum_\alpha h^r_{\alpha\alpha} = \sum_t h^r_{tt} = 0, \tag{3.17}$$

for $r = n+2, \dots, 2m$.

Finally, by applying (3.6) and (3.16), we obtain

$$\frac{n_2\Delta f}{f} = \frac{n^2}{4}H^2 + n_1 n_2 c + 3c\sum_{\alpha,t}\langle Pe_\alpha, e_t\rangle^2 \le \frac{n^2}{4}H^2 + n_1 n_2 c \tag{3.18}$$

which implies inequality (1.1).

If the equality sign of (1.1) holds, then all of the inequalities in (3.12), (3.16) and (3.18) become equalities. Hence, we obtain (3.13), (3.17) and the equation:

$$\langle Pe_\alpha, e_t\rangle = 0.$$

Therefore, we have Conditions (1), (2) and (3) of Theorem 1.

Conversely, if Conditions (1), (2) and (3) of Theorem 1 hold, then the equality case of inequalities in (3.15), (3.16) and (3.18) become equalities. Hence, we obtain the equality case of (1.1). This proves Theorem 1.

If $\phi : N_1 \times_f N_2 \to CH^m(4c)$ is an isometric minimal immersion of a warped product $N_1 \times_f N_2$ into the complex hyperbolic m-space, then Theorem 1 implies that

$$\frac{\Delta f}{f} \le n_1 c < 0. \tag{3.19}$$

Thus, f cannot be a harmonic function or an eigenfunction of Laplacian with positive eigenvalue. This proves Theorems 2 and 3.

Since the warping function f is positive, (3.19) implies that $\Delta f < 0$. Thus, if N_1 is compact, the warping function f must be constant by applying Hopf's lemma which contradicts to Theorem 2. Hence, we obtain Theorem 4. □

4. Additional results. A submanifold N in CH^m is called *totally real* if $J(TN) \subset T^\perp N$ (see [4]).

When $\dim N_1 = \dim N_2 = 1$, Theorem 1 implies immediately the following:

Corollary 1. *If* $\dim N_1 = \dim N_2 = 1$*, then an isometric immersion of a warped product* $N_1 \times_f N_2$ *into* $CH^m(4c)$ *satisfies the equality case of inequality* (1.1) *if and only if it is a totally real totally umbilical surface.*

By applying Theorem 1 we also have

Corollary 2. *If* $\dim N_1 = \dim N_2$*, then the warping function* f *of every warped product decomposition* $N_1 \times_f N_2$ *of a real space form is an eigenfunction of the Laplacian.*

Proof. Assume that $N_1 \times_f N_2$ is a warped product decomposition of a real space form $R^{2n_1}(\varepsilon)$ of constant curvature ε with $\dim N_1 = \dim N_2 = n_1$. Let c be a negative number $< \varepsilon$. Then locally there is a totally umbilical isometric immersion j of $N_1 \times_f N_2$ into the real hyperbolic space $H^{2n_1+1}(c)$ of constant curvature c.

Denote by

$$\iota : H^{2n_1+1}(c) \to CH^{2n_1+1}(4c)$$

the standard totally real totally geodesic isometric imbedding of $H^{2n_1+1}(c)$ into $CH^{2n_1+1}(4c)$. Then the composition:

$$\phi = \iota \circ j : N_1 \times_f N_2 \xrightarrow{\text{totally umbilical}} H^{2n_1+1}(c) \xrightarrow[\text{totally real}]{\text{totally geodesic}} CH^{2n_1+1}(4c)$$

is an isometric immersion which satisfies Conditions (1), (2) and (3) of Theorem 1. Hence, ϕ satisfies the equality case of (1.1) according to Theorem 1. Therefore, we have

$$\frac{\Delta f}{f} = n_1 H^2 + n_1 c.$$

Since the composition ϕ is a totally real, totally umbilical isometric immersion, it has constant squared mean curvature. Thus, the warping function f is an eigenfunction of the Laplacian. □

Definition 1. Let $\psi : N_1 \times_f N_2 \to M$ be an isometric immersion of a warped product into a Riemannian manifold. Then ψ is called *pseudo umbilical* if the shape operator $A_{\overrightarrow{H}}$ at the mean curvature vector satisfies $A_{\overrightarrow{H}}X = \lambda X$ for some function λ, where X is an arbitrary vector tangent to $N_1 \times_f N_2$.

The immersion is called N_j*-pseudo umbilical* if the shape operator $A_{\overrightarrow{H}}$ satisfies $A_{\overrightarrow{H}}X = \lambda X$ for tangent vectors X in $\mathcal{D}_j$ ($j = 1$ or 2).

For warped products satisfying the equality case of (1.1), we also have

Proposition 1. *Let $\phi : N_1 \times_f N_2 \to CH^m(4c)$ be an isometric immersion of a warped product $N_1 \times_f N_2$ into the complex hyperbolic m-space $CH^m(4c)$. If ϕ satisfies the equality case of* (1.1)*, then we have*:

(i) *$\langle h(X, Y), JZ\rangle = 0$ for tangent vectors X, Y in $\mathcal{D}_1$ and Z in $\mathcal{D}_2$.*

(ii) *ϕ is N_2-pseudo umbilical.*

(iii) *If $\dim N_1 \neq \dim N_2$, then ϕ is non-pseudo umbilical.*

(iv) *If f in non-constant, then we have $J\mathcal{D}_2 \neq \mathcal{D}_2$,* i.e.*, $\mathcal{D}_2$ is non-holomorphic.*

Proof. Assume that $\phi : N_1 \times_f N_2 \to CH^m(4c)$ is an isometric immersion which satisfies the equality case of (1.1). Then Conditions (1), (2) and (3) of Theorem 1 holds. Let X, Y be vector fields in $\mathcal{D}_1$ and Z in $\mathcal{D}_2$. Then we have $\nabla_X Y \in \mathcal{D}_1$ by (3.1). Thus, by applying Condition (3) of Theorem 1, we have $\langle J\nabla_X Y, Z\rangle = 0$. Hence, by applying (3.1) and Conditions (1) and (3) of Theorem 1, we obtain

$$\begin{aligned} 0 &= X \langle JY, Z\rangle \\ &= \langle Jh(X, Y), Z\rangle + \langle JY, \nabla_X Z\rangle \\ &= \langle Jh(X, Y), Z\rangle , \end{aligned} \tag{4.1}$$

which implies statement (i).

From (3.3), (3.4) and Conditions (1) and (3) of Theorem 1, we have

$$\langle\, \operatorname{trace} h_1, h(Z, Z)\,\rangle = \frac{\Delta f}{f} - n_1 c \tag{4.2}$$

for any unit tangent vector Z in $\mathcal{D}_2$. Therefore, by applying polarization, we find

$$\langle\, \operatorname{trace} h_1, h(Z, W)\,\rangle = 0 \tag{4.3}$$

for orthonormal vectors Z, W in $\mathcal{D}_2$.

On the other hand, Condition (2) of Theorem 1 implies that

$$\operatorname{trace} h_1 = \frac{n}{2}\overrightarrow{H}.$$

Hence, by (4.2), (4.3) and Condition (1) of Theorem 1, we obtain

$$A_{\overrightarrow{H}} Z = \frac{2}{n}\left\{\frac{\Delta f}{f} - n_1 c\right\} Z$$

for tangent vector Z in $\mathcal{D}_2$. Hence, ϕ is a N_2-pseudo umbilical immersion. This proves statement (ii).

Statement (iii) follows easily from Condition (2) of Theorem 1.

Let $\hat{h}$ and $\hat{A}$ denote the second fundamental form and shape operator of N_2 in $N_1 \times_f N_2$. Then, by applying (3.1), we have

$$\begin{aligned} \left\langle \hat{h}(Z, W), X\right\rangle &= \langle \nabla_Z W, X\rangle \\ &= -\langle W, \nabla_Z X\rangle \\ &= -(X \ln f)\, \langle Z, W\rangle \end{aligned} \tag{4.4}$$

for tangent vector fields X in $\mathcal{D}_1$ and Z, W in $\mathcal{D}_2$.

If $\mathcal{D}_2$ is a holomorphic distribution, then each fiber is immersed in $CH^m(4c)$ as a holomorphic submanifold. Hence, we obtain $\operatorname{trace} \hat{h} = 0$. Thus, from (4.4) we conclude that the warping function f is constant. □

Remark. In views of Theorem 2 and Theorem 3, it is interesting to point out that there do exist isometric minimal immersions from warped products into complex hyperbolic spaces such that the warping functions of the warped products are eigenfunctions of Laplacian with negative eigenvalue.

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