A Laplacian comparison theorem and its applications

By Takeshi Kura

Department of Mathematics, Graduate School of Science, Hiroshima University,
1-3-1, Kagamiyama, Higashi-Hiroshima, Hiroshima 739-8526

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Abstract: A Laplacian comparison theorem is given. As applications we show a volume comparison theorem and a criterion for the hyperbolicity of Riemannian manifolds.

Key words: Laplacian comparison theorem; volume comparison theorem; hyperbolicity of Riemannian manifolds.

1. Introduction. Let M be a smooth connected complete Riemannian n-manifold, $n \geq 2$, without boundary. Let P be a fixed point in M and define $h(x) = d(x, \mathrm{P})$ for all $x \in M$, where d denotes the geodesic distance. Let K_M and Ric_M denote the sectional curvature and the Ricci curvature of M, respectively. Let $0 < l \leq \infty$ and $\gamma : [0, l) \to M$ be a minimal geodesic with $\gamma(0) = \mathrm{P}$, $|\gamma'(0)| = 1$. Let $k, r : [0, \infty) \to \mathbf{R}$ be two continuous functions. We assume that k satisfies

$$\mathrm{K}_M(\gamma'(t), X) \leq k(t), \tag{1}$$

for $\forall t \in (0, l)$, $\forall X \in M_{\gamma(t)}$, $X \perp \gamma'(t)$. Let f be a solution of

$$\begin{cases} f'' + k(t)f = 0, \ f(t) > 0, \ (0 < t < l), \\ f(0) = 0, \ f'(0) = 1. \end{cases} \tag{2}$$

The Hessian comparison theorem (cf. Kasue [6, Lemma 2.18]) shows that

$$\Delta h(\gamma(t)) \geq (n-1)\frac{f'(t)}{f(t)}, \quad \forall t \in (0, l). \tag{3}$$

The purpose of this note is to improve the above inequality. We see from (1) that

$$\mathrm{Ric}_M(\gamma'(t), \gamma'(t)) \leq (n-1)k(t).$$

We impose the assumption that

$$\mathrm{Ric}_M(\gamma'(t), \gamma'(t)) \leq r(t) \leq (n-1)k(t), \tag{4}$$

for $\forall t \in (0, l)$. Let f_1 be a solution of

$$\begin{cases} f_1'' + \{r(t) - (n-2)k(t)\}f_1 = 0, \quad (0 < t < l), \\ f_1(0) = 0, \ f_1'(0) = 1. \end{cases} \tag{5}$$

Our main result is the following

Theorem 1. *If $f'(t) \geq 0$ on $(0, l)$, then*

$$f_1(t) \geq f(t), \ \frac{f_1'(t)}{f_1(t)} \geq \frac{f'(t)}{f(t)}, \quad \forall t \in (0, l). \tag{6}$$

$$\Delta h(\gamma(t)) \geq (n-2)\frac{f'(t)}{f(t)} + \frac{f_1'(t)}{f_1(t)}, \quad \forall t \in (0, l). \tag{7}$$

Theorem 1 also generalizes the inequality of Borbély ([1, Lemma 2]), which is the motivation of this note. As applications of Theorem 1 we obtain the volume comparison theorem – Theorem 2 – and a criterion for the hyperbolicity of Riemannian manifolds, i.e., the existence of the Green's function of Laplacian – Theorem 3 and Theorem 4.

2. Proof of Theorem 1. We need the following lemma.

Lemma 1 ([1, Proposition 4]). *Let $n \in \mathbf{N}$, $a \geq 0$, and $b \geq na^2$. Let S be the set*

$$\left\{(x_1, \ldots, x_n) \in \mathbf{R}^n \;\middle|\; a \leq x_1 \leq \cdots \leq x_n, \sum_{j=1}^{n} x_j^2 \geq b\right\}.$$

Define $f : S \to \mathbf{R}$ by $f(x_1, \ldots, x_n) = x_1 + \cdots + x_n$. Then

$$\min f(S) \geq (n-1)a + \{b - (n-1)a^2\}^{1/2}.$$

We will follow Chavel's notation [2, pp. 63–67] as in [1] and [8]. Let $v = \gamma'(0)$, $M_t^{\perp}$ denote the orthogonal complement of $\gamma'(t)$ in $M_{\gamma(t)}$, and define $R(t) : M_t^{\perp} \to M_t^{\perp}$ by $R(t)X = R(\gamma'(t), X)\gamma'(t)$, where $R(\ , \ )$ is the curvature tensor of M. Let $\tau_t : M_p \to M_{\gamma(t)}$ be the parallel translation along γ and define $\mathcal{R}(t) : v^{\perp} \to v^{\perp}$ by $\mathcal{R}(t)X = (\tau_t)^{-1}R(t)\tau_t(X)$. Let A be the solution of

$$A'' + \mathcal{R}A = 0 \text{ on } (0, l), \ A(0) = 0, \ A'(0) = I.$$

Let g_{ij} be the components of the Riemannian metric of M with respect to the normal coordinate system

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around P, $U(t) = A'(t)A(t)^{-1}$, and define $g(t,w) = t^{2(n-1)} \det g_{ij} \circ \exp(tw)$. We note that U and R are selfadjoint and that g, A, U, and $\mathcal{R}$ have the following properties:

$$\sqrt{g(t,v)} = \det A(t). \tag{8}$$

$$U' + U^2 + \mathcal{R} = 0 \text{ on } (0,l). \tag{9}$$

$$\operatorname{tr} U(t) = \frac{d}{dt} \log \det A(t) = \Delta h(\gamma(t)). \tag{10}$$

$$\operatorname{tr} \mathcal{R}(t) = \operatorname{Ric}_M(\gamma'(t), \gamma'(t)). \tag{11}$$

We see from Sturm's comparison theorem that $f_1(t) > 0$ on $(0,l)$. Let $F(t) = f'(t)/f(t)$ and $F_1(t) = f_1'(t)/f_1(t)$. Then the Riccati equations

$$F' + F^2 + k = 0, \quad F_1' + F_1^2 + r - (n-2)k = 0$$

hold on $(0,l)$. Since $\lim_{t\to+0}(F_1(t) - F(t)) = 0$ we infer from [4, Theorem] that $F_1(t) \geq F(t)$ on $(0,l)$, which implies $f_1(t) \geq f(t)$ on $(0,l)$ because $\lim_{t\to+0} f_1(t)/f(t) = 1$. Let $v(t) = \operatorname{tr} U(t)$ and $\alpha_1, \ldots, \alpha_{n-1}$ be the eigenvalues of U. From (9) we have

$$v' + \alpha_1^2 + \cdots + \alpha_{n-1}^2 + \operatorname{tr} \mathcal{R} = 0.$$

We see from [4, Theorem] that $\alpha_j(t) \geq F(t) \geq 0$ $(1 \leq j \leq n-1)$. Applying Lemma 1 we find that

$$\begin{aligned} v &\geq (n-2)F + \sqrt{\alpha_1^2 + \cdots + \alpha_{n-1}^2 - (n-2)F^2} \\ &= (n-2)F + \sqrt{-v' - \operatorname{tr}\mathcal{R} - (n-2)F^2}. \end{aligned}$$

Let $V = v - (n-2)F$. Then we have

$$\begin{aligned} V' + V^2 &\geq -\operatorname{tr}\mathcal{R} + (n-2)k \\ &\geq -r + (n-2)k \\ &= F_1' + F_1^2. \end{aligned}$$

Since $\liminf_{t\to+0}(V(t) - F_1(t)) \geq 0$ and $V(t) + F_1(t) \geq 0$, we conclude from [4, Theorem] that $V(t) \geq F_1(t)$ on $(0,l)$ and the proof of Theorem 1 is complete.

3. Applications. Let $B(t) = \{x \in M \mid d(x,\mathrm{P}) < t\}$ and $\operatorname{vol}(B(t))$ be the volume of $B(t)$. We have the following volume comparison theorem.

Theorem 2. *Let $f'(t) \geq 0$ on $(0,l)$ and $T = \min\{l$, the injectivity radius of P$\}$. Assume that k and r satisfy*

$$\mathrm{K}_M(\gamma'(t), X) \leq k(t),$$

and

$$\operatorname{Ric}_M(\gamma'(t), \gamma'(t)) \leq r(t) \leq (n-1)k(t)$$

for any minimal geodesic $\gamma : [0,T) \to M$ with $\gamma(0) = \mathrm{P}$ and $|\gamma'(0)| = 1$, $\forall t \in (0,T)$, and $\forall X \in M_{\gamma(t)}$ with $X \perp \gamma'(t)$. Then

$$(0,T) \ni t \longmapsto \operatorname{vol}(B(t)) \Big/ \int_0^t f_1(s) f(s)^{n-2} ds$$

is a nondecreasing function.

Proof. We will follow Chavel's notation [3, p. 107]. Let g_{ij} be as in the proof of Theorem 1 and H_{n-1} be the $(n-1)$-dimensional Hausdorff measure of M_p. We define

$$F(t) = t^{n-1} \int_{S(\mathrm{P};1)} \sqrt{\det g_{ij} \circ \exp(tx)} dH_{n-1}(x)$$

for $0 < t < T$. From Theorem 1 we have

$$\begin{aligned} & t^{n-1} \sqrt{\det g_{ij} \circ \exp(tx)} \Big/ f_1(t) f(t)^{n-2} \\ & \quad \geq s^{n-1} \sqrt{\det g_{ij} \circ \exp(sx)} \Big/ f_1(s) f(s)^{n-2} \end{aligned}$$

for $0 < s < t < T$, $x \in S(\mathrm{P};1)$. Therefore $F/f_1 f^{n-2}$ is nondecreasing on $(0,T)$. The Theorem follows by applying [3, Lemma 3.1]. □

We assume that $l = \infty$ in (2) and (5) and that the injectivity radius of P is infinity. Let ω_{n-1} be the volume of the unit $(n-1)$-sphere in $\mathbf{R}^n$. The following theorem improves [5, Theorem 2.2] and [6, Theorem 5.3] in case $f' \geq 0$, whose condition holds if $k \leq 0$ or $k \geq 0$.

Theorem 3. *Let $f'(t) \geq 0$ on $(0,\infty)$. Assume that k and r satisfy*

$$\mathrm{K}_M(\gamma'(t), X) \leq k(t),$$

and

$$\operatorname{Ric}_M(\gamma'(t), \gamma'(t)) \leq r(t) \leq (n-1)k(t)$$

for any geodesic $\gamma : [0,\infty) \to M$ with $\gamma(0) = \mathrm{P}$ and $|\gamma'(0)| = 1$, $\forall t \in (0,\infty)$, and $\forall X \in M_{\gamma(t)}$ with $X \perp \gamma'(t)$. If

$$\int_T^\infty \frac{dt}{f_1(t) f(t)^{n-2}} < \infty$$

for some $T > 0$, then the Green's function $G(x,\mathrm{P})$ of M with a pole at P exists and satisfies

$$G(x,p) \leq \frac{1}{\omega_{n-1}} \int_{h(x)}^\infty \frac{dt}{f_1(t) f(t)^{n-2}}$$

for all $x \in M$, $x \neq \mathrm{P}$.

Proof. We define

$$F(t) = \omega_{n-1}^{-1} \int_t^\infty f_1(s)^{-1} f(s)^{2-n} ds$$

for $t > 0$. From Theorem 1 we see that

$$\Delta F(h(x)) = F''(h(x)) + F'(h(x))\Delta h(x) \leq 0$$

for $x \in M \setminus \{\mathrm{P}\}$. We can prove the theorem after the proof of [6, Theorem 4.3]. □

Let $1 < \alpha \leq n$ be a constant. We define k_α and c_2 as in [7]. The following theorem improves [7, Theorem 1].

Theorem 4. *Let $f'(t) \geq 0$ on $(0, \infty)$. Assume that k and r satisfy*

$$\mathrm{K}_M(\gamma'(t), X) \leq k(t),$$

and

$$\mathrm{Ric}_M(\gamma'(t), \gamma'(t)) \leq r(t) \leq (n-1)k(t)$$

for any geodesic $\gamma : [0, \infty) \to M$ with $\gamma(0) = \mathrm{P}$ and $|\gamma'(0)| = 1$, $\forall t \in (0, \infty)$, and $\forall X \in M_{\gamma(t)}$ with $X \perp \gamma'(t)$. If

$$\int_T^\infty f_1(t)^{-1/(\alpha-1)} f(t)^{(2-n)/(\alpha-1)} dt < \infty$$

for some $T > 0$, then the α-Green's function $G(x, \mathrm{P})$ of M with a pole at P exists and satisfies

$$G(x, \mathrm{P}) \leq k_\alpha c_2 \int_{h(x)}^\infty f_1(t)^{-1/(\alpha-1)} f(t)^{(2-n)/(\alpha-1)} dt$$

for all $x \in M$, $x \neq \mathrm{P}$.

Proof. We define

$$F(t) = \int_t^\infty f_1(s)^{-1/(\alpha-1)} f(s)^{(2-n)/(\alpha-1)} ds$$

for $t > 0$. Let $u(x) = F(d(x))$, then

$$\begin{aligned} &\mathrm{div}(|\nabla u|^{\alpha-2}\nabla u) \\ &= -|F(d)|^{\alpha-2}(|F'(d)|\Delta h - (\alpha-1)F''(d)) \leq 0 \end{aligned}$$

in $M \setminus \{\mathrm{P}\}$. We can prove the theorem after the proof of [7, Theorem 1]. □

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