Complex contact three manifolds with Legendrian vector fields

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Abstract: We classify the local structure of complex contact manifolds of dimension three with Legendrian vector fields. We also give the partial classification of the complex contact manifolds with Legendrian vector fields, in case the manifolds admit holomorphic contact 1-forms.

Key words: Complex contact manifold; complex vector field; Anosov action.

1. Introduction. A complex contact manifold is a closed complex manifold M of dimension $2n+1$ with an open atlas $\mathcal{U} = \{U\}$ such that on each open set U there is a holomorphic 1-form ω satisfying $\omega \wedge (d\omega)^n \neq 0$ and on the intersection $U \cap U'$ there exists a holomorphic function $f\colon U \cap U' \to \mathbb{C}^*$ such that $\omega = f\omega'$. In particular, $\xi = \bigcup_{U \in \mathcal{U}} (\ker \omega)$ is a well-defined holomorphic sub-bundle of $\mathrm{T}^{1,0}M$. We denote this complex contact manifold by (M, ξ).

A nonsingular Legendrian vector field of (M, ξ) is a holomorphic section of $\xi \setminus \{0\}$, where $\{0\}$ is the zero-section of ξ. In other words, X is a nonvanishing holomorphic vector field satisfying $\omega(X) = 0$ on each open set U.

Let us investigate deformations of contact structures. Suppose that $\{\xi_t\}$ ($t \in U \subset \mathbb{C}$) is a holomorphic family of contact structures. On a neighborhood U of each point $p \in M$, there is a holomorphic vector field $\widetilde{X}$ on U such that X is Legendrian to ξ_0 and $(\partial/\partial s)(\xi^s|_p)|_{s=0}$ coincides with $(\partial/\partial t)(\xi_t|_p)|_{t=0}$, where ξ^s is a push-forward of ξ_0 by the action ϕ^s which is generated by the vector field $\widetilde{X}$. The choice of $\widetilde{X}$ is not unique, but the vector $\widetilde{X}_p$ on the point p is uniquely determined. Let X be the vector field such that on each point p it determines the vector $\widetilde{X}_p$. The vector X is holomorphic and Legendrian to ξ_0. The condition $X_p \neq 0$ corresponds to $(\partial/\partial t)(\xi_t|_p)|_{t=0} \neq 0$. Thus the Legendrian vector fields correspond to the 1-jet of the holomorphic non-degenerate deformations of the contact structures.

The purpose of this paper is to investigate complex contact manifolds of dimension three with Legendrian vector fields. In the section 2, we classify the local structures of Legendrian vector fields into two types, namely parabolic type and loxodromic type.

In the section 3, we first introduce a class of examples that are defined on the homogeneous spaces of 3-dimensional Lie groups. We also give some examples that do not admit any global contact form.

In the section 4, we investigate the special case so that the contact manifold admits a global contact form.

2. Local structure. We obtain the following theorem which classifies the local structures of Legendrian vector fields into two types.

Theorem 1. *Suppose (M, ξ) is a connected complex contact manifold of complex dimension three and X is a Legendrian vector field of (M, ξ). Then (M, ξ, X) has either parabolic* (P) *or loxodromic* (L) *local structure*;

(P) *There is an open atlas $\mathcal{U} = \{U = \{(x, y, z)\}\}$ such that on each coordinated open set U, $\xi = \ker(dy - x dz)$ and $X = \partial/\partial x$.*

(L) *There is an open atlas $\mathcal{U} = \{U = \{(x, y, z)\}\}$ such that on each coordinated open set U, $\xi = \ker(dy - \exp(ax) dz)$ and $X = \partial/\partial x$, where $a \in \mathbb{C}^*$ is a constant, which does not depend on the choice of U.*

Sketch of the proof. Let $\phi\colon \mathbb{C} \times M \to M$; $(t, p) \mapsto \phi^t(p)$ be the holomorphic $\mathbb{C}$-action generated by X.

The projectivized bundle $\mathrm{P}X$ is a $\mathbb{C}\mathrm{P}^1$-bundle over M, so that for any $p \in M$ the fiber P_pX is the set of complex 2-dimensional subspaces ν_p of T_pM that satisfy $X_p \in \nu_p$. The holomorphic $\mathbb{C}$-action ϕ induces a holomorphic $\mathbb{C}$-action Φ on the bundle $\mathrm{P}X$.

It is shown that the $\mathbb{C}\mathrm{P}^1$-bundle $\mathrm{P}X$ has a unique trivialization $I\colon \mathrm{P}X \to M \times \mathbb{C}\mathrm{P}^1$ up to the congugacies of $\mathrm{Aut}(\mathbb{C}\mathrm{P}^1)$ and the action Φ on $\mathrm{P}X \approx$

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$M \times \mathbb{C}\mathrm{P}^1$ splits into the action ϕ on M and an action ψ on $\mathbb{C}\mathrm{P}^1$.

A holomorphic $\mathbb{C}$-action on $\mathbb{C}\mathrm{P}^1$ is either the null action ($x \mapsto x$), the parabolic action (conjugate to $x \mapsto x+t$) or the loxodromic action (conjugate to $x \mapsto \exp(at)x$). The action of ψ is not null because ξ is contact.

If the action ψ is parabolic, it induces the case (P) and if loxodromic, the case (L). □

3. Examples. We first introduce a class of examples that are defined on the homogeneous spaces of the 3-dimensional complex Lie groups. Since the complex Lie groups are easily classified, we can classify such examples. Here we give the complete classification of these spaces. It is an important fact that the contact structures on the homogeneous spaces are defined by a global contact form ω on M.

Example 2. Let $\mathrm{SL}(2,\mathbb{C})$ be the Lie group of the 2×2 complex matrices with determinants 1. The space of the left invariant vector fields of $\mathrm{SL}(2,\mathbb{C})$ forms a Lie algebra $sl(2,\mathbb{C})$ of the 2×2 complex traceless matrices. Let us set

$$T = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \ U = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

They form a basis of $sl(2,\mathbb{C})$ and satisfy

$$[T,U] = -U, \quad [T,S] = S, \quad [U,S] = -2T.$$

Let Γ be a cocompact discrete subgroup of $\mathrm{SL}(2,\mathbb{C})$ and $M = \Gamma \backslash \mathrm{SL}(2,C)$. The vector fields T, U and S are also well-defined in M.

Example 2-1 (geodesic). Let a and b be non-zero complex numbers. Define a 1-form ω and a vector field X by $\omega(T) = 0$, $\omega(U) = \omega(S) = b$ and $X = aT$.

Example 2-2 (horocycle). Let a and b be non-zero complex numbers. Define a 1-form ω and a vector field X by $\omega(T) = b$, $\omega(U) = \omega(S) = 0$ and $X = aU$.

Example 3. Let $\mathrm{Solv} = \{(x,y,z) \mid x,y,z \in \mathbb{C}\}$ be the Lie group defined by

$$(a,b,c)\cdot(x,y,z) = (x+a, e^{-a}y+b, e^{a}z+c).$$

The left invariant vector fields

$$H = \partial/\partial x, \quad U = e^{-x}\partial/\partial y, \quad S = e^{x}\partial/\partial z$$

satisfy

$$[H,U] = -U, \quad [H,S] = S, \quad [U,S] = 0$$

and span the Lie algebra *solv*. Let Γ be a cocompact subgroup of Solv and $M = \Gamma\backslash\mathrm{Solv}$. The vector fields H, U and S are also well-defined in M.

Example 3-1. Let $a \in \mathbb{C}$ be a non-zero complex number. Define a 1-form ω and a vector field X by $\omega(H) = 0$, $\omega(U) = \omega(S) = 1$ and $X = aH$.

Example 3-2. Let $a \in \mathbb{C}$ be a non-zero complex number. Define a 1-form ω and a vector field X by $\omega(H) = 0$, $\omega(U) = \omega(S) = 1$ and $X = (a/2)(U + S)$.

Example 4. The Lie algebra *nil* of the Lie group

$$\mathrm{Nil} = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| \ x,y,z \in \mathbb{C} \right\}$$

is spanned by

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

They satisfy

$$[P,Q] = 0, \quad [P,R] = Q, \quad [Q,R] = 0.$$

Let Γ be a cocompact discrete subgroup of Nil and $M = \Gamma\backslash\mathrm{Nil}$. P, Q and R are also well-defined in M. Define a 1-form ω and a vector field X by $\omega(P) = \omega(R) = 0$, $\omega(Q) = 1$ and $X = P$.

We also introduce some examples that do not belong to the class of homogeneous spaces. The following examples do not allow the global contact form, so they are not homogeneous spaces.

Example 5. Suppose a matrix $A = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{C})$ preserves a lattice $\Lambda \subset \mathbb{C}^2$ and $|\lambda_1|, |\lambda_2| \neq 1$. Let $\bar{A}$ be the corresponding diffeomorphism of a complex torus $N = \mathbb{C}^2/\Lambda$. Choose $n \in \mathbb{Z}$ and $\eta \in \mathbb{C}$ so that $\exp(n\eta) = \lambda_2/\lambda_1$ and $\exp(m\eta) \neq \lambda_1$ for any $m \in \mathbb{Z}$. Define a manifold $M = \mathbb{C} \times N/\!\sim$, where $\sim$ is an equivalence relation

$$(x,[y,z]) \sim (x-\eta, \bar{A}([y,z])) \sim (x - 2\pi i, [y,z]).$$

The vector field $X = \partial/\partial x$ is well-defined on M. The universal cover $\widetilde{M}$ of M is isomorphic to $\mathbb{C}^3$. Let $\widetilde{\omega} = dy - \exp(nx)dz$ be a holomorphic contact structure on $\widetilde{M}$. The contact structure $\xi = \ker\widetilde{\omega}$ is well-defined on M, thus X is a Legendrian flow of the contact manifold (M,ξ).

Example 6. Let a^+, a^-, λ and η be non-zero complex numbers and n be a positive integer satisfying that $\mathrm{Re}\,\eta > 0$, $|\lambda| > 1$ and $a^-/a^+ = \exp(n\eta)$.

Let

$$\widetilde{M} = \{(x, y, z) \in \mathbb{C}^3 \mid (y, z) \neq (0, 0)\}$$
$$= \mathbb{C} \times (\mathbb{C}^2 \setminus \{(0, 0)\})$$

and define a manifold M by $\widetilde{M}/\sim$, where $\sim$ is an equivalence relation defined by

$$(x, y, z) \sim (x, \lambda y, \lambda z) \sim (x - \eta, a^+ y, a^- z).$$

The vector field $X = \partial/\partial x$ and the contact structure $\xi = \ker(dy - \exp(nx)dz)$ on $\widetilde{M}$ are well-defined on M, thus X is a Legendrian vector field of (M, ξ).

4. If there is a global contact form. In this section, we suppose that the contact structure ξ is defined by a global holomorphic 1-form ω on M. We denote such a contact manifold by (M, ω). Examples of homogeneous spaces given in the section 3 satisfy this condition and we do not know other examples. We are going to give a partial classification.

The Reeb vector field of the contact form ω is the vector field Y on M satisfying $\omega(Y) = 1$ and $Y \rfloor d\omega = 0$. The Reeb vector field Y of ω is uniquely defined and holomorphic.

Proposition 7. *The Legendrian vector field X and the Reeb vector field Y satisfy one of the following conditions*;

(i) $[X, Y] = Z$ *and* $\mathrm{T}M = \mathbb{C}X \oplus \mathbb{C}Y \oplus \mathbb{C}Z$.
(ii) $[X, Y] = cX$, *where* $c \neq 0$.
(iii) $[X, Y] = 0$.

The following remark is very important so that it is used repeatedly to prove propositions and lemmas throughout this paper, and this is the reason why our theory does not work on other (for example, real analytic) categories.

Remark 8. Since M is compact, any holomorphic map $f\colon M \to \mathbb{C}$ is a constant map because of the maximum principal.

In the following subsections, we investigate each cases. We use the following lemma to investigate the case (ii) and (iii).

Lemma 9. *For any holomorphic action $f\colon \mathbb{C} \times M \to M$, the non-vanishing $(3, 0)$-form $\omega \wedge d\omega$ is invariant under the action of f.*

4.1. Case (i); $[X, Y] = Z$. In this case, it is trivial that M is a homogeneous space. We can show that (M, ω, X) is either the Example 3-1 or 2-1 by some calculation.

4.2. Case (ii); $[X, Y] = cX$. Normalizing ω by $c\omega$, we can suppose that $[X, Y] = X$. Étienne Ghys and Alberto Verjovsky showed the following proposition by using Anosov property of the action generated by Y.

Proposition 10 ([GV]). *Suppose that X and Y are non-vanishing holomorphic vector fields on a closed complex 3-manifold M, the bracket $[X, Y]$ equals to X and both actions generated by X and Y preserve a volume on M. Then there uniquely exists a holomorphic line field E^s such that it is transverse to the plane field $\mathbb{C}X \oplus \mathbb{C}Y$ and invariant under the action generated by Y.*

We can define a holomorphic vector field Z by $E^s = \mathbb{C}Z$ and $\omega \wedge d\omega(X, Y, Z) = 1$, so M is a homogeneous space. This is the Example 2-2.

4.3. Case (iii); $[X, Y] = 0$. If M is a homogeneous space, it is either the Example 4 or 3-2.

Let us consider the general case. We begin by investigating the local structure.

Proposition 11. *There is an open atlas $\mathcal{U} = \{U = \{(x, y, z)\}\}$ such that on each open set U, $\omega = dy - xdz$, $X = \partial/\partial x$ and $Y = \partial/\partial y$.*

This proposition is proved by using the Lemma 9 and the Theorem 1.

Let us consider the global structure. Define a 1-form α by $\alpha = \omega - {\phi^1}^*(\omega)$. By the Proposition 11, $\alpha = dz$ on each U.

Fix a point $p \in M$. Let Ω be the set of paths $\gamma\colon [0, 1] \to M$ such that $\gamma(0) = p$. Obviously, the universal cover $\widetilde{M} = \Omega/\text{homotopy}$. Let us define a map $\pi\colon \widetilde{M} \to \mathbb{C}$ by $\pi([\gamma]) = \int_\gamma \alpha$. It is shown that this map is surjective, defines $\mathbb{C}^2$-bundle structure on $\widetilde{M}$ and $\mathrm{T}F = \mathbb{C}X \oplus \mathbb{C}Y$ for each fiber F.

4.4. Conclusion. Pulling these three cases together, we have the following theorem.

Theorem 12. *Suppose X is a Legendrian flow of a complex contact manifold (M, ω), then either it is given in the Examples* 2-1, 2-2 *or* 3-1; *or it has the local structure $\omega = dy - xdz$, $X = \partial/\partial x$ and $\widetilde{M}$ is a $\mathbb{C}^2$-bundle over $\mathbb{C}$.*

4.5. Some remarks on the case (iii) if $\widetilde{M}$ is a trivial bundle. Suppose that $\widetilde{M} \approx \mathbb{C}^3$, $\omega = dy - xdz$ and $X = \partial/\partial x$. We obtain the following proposition by the straightforward calculation.

Proposition 13. *The group* $\mathrm{Aut}(\mathbb{C}^3, \omega, X)$ *of automorphisms on $\mathbb{C}^3$ that preserve $\omega = dy - xdz$ and $X = \partial/\partial x$ is given by*

$$\mathrm{Aut}(\mathbb{C}^3, \omega, X) = \{(f, C) \mid f \in \mathcal{O}(\mathbb{C}),\ C \in \mathbb{C}\},$$
$$(f, C)\colon (x, y, z) \mapsto (x + f'(z), y + f(z), z + C).$$

The classification of M corresponds to the classification of subgroups $\Gamma \subset \mathrm{Aut}(\mathbb{C}^3, \omega, X)$ such that $\Gamma \backslash \mathbb{C}^3 \approx M$. We obtain the result that any elements of such a subgroup can be represented by exponentials and polynomials up to conjugate classes.

Theorem 14. *Suppose that Γ is a subgroup of $\mathrm{Aut}(\mathbb{C}^3, \omega, X)$ such that $\Gamma \backslash \mathbb{C}^3$ is a closed 3-dimensional complex manifold. Then there exist an element $\sigma \in \mathrm{Aut}(\mathbb{C}^3, \omega, X)$, an integer $N \leq 16$ and complex numbers $\alpha_1, \dots, \alpha_N$ such that for any $(f, C) \in \sigma^{-1}\Gamma\sigma$, the function f is written as*

$$f(z) = \sum_{k=1}^{N} r_k(z) \exp(\alpha_k z) + b_0,$$

where r_k is a polynomial of degree 4 or less and b_0 is a constant.

To prove the theorem above, we use the following lemmas.

Lemma 15 ([Gu]). *For any holomorphic function $f : \mathbb{C} \to \mathbb{C}$ and non-zero complex number η, there exists a holomorphic function $g : \mathbb{C} \to \mathbb{C}$ such that $g(x + \eta) - g(x) = f(x)$.*

Lemma 16. *Let $\eta_1\mathbb{Z} \oplus \eta_2\mathbb{Z}$ be a lattice of $\mathbb{C}$. If (f_1, η_1), (f_2, η_2) and $(f, 0) \in \Gamma$, the function f is denoted by*

$$f(z) = \sum_{k=1}^{N} p_k(z) \exp(\alpha_k z),$$

where p_k is a polynomial of degree 3 or less, N is an integer less than or equal to 16 and $\alpha_k \in \mathbb{C}$.

Example 17. Let $\Gamma = \{(az + b, c) \mid a, b, c \in \mathbb{Z}[\sqrt{-1}]\}$. This is given in the Example 4.

Example 18. Let n_1, n_2 be integers, such that $|n_1| < 2 < |n_2|$; α_j, β_j be the solutions of the quadratic equations $t^2 - n_j t + 1 = 0$ ($j = 1, 2$) and η_1, η_2 be complex numbers such that $\exp(\eta_1) = \alpha_1$, $\exp(\eta_2) = \alpha_2$ and $\eta_1\mathbb{Z} \oplus \eta_2\mathbb{Z}$ is a lattice of $\mathbb{C}$.

Define four functions $f_0, \dots, f_3$ by

$$\begin{aligned} f_0(z) &= e^z + e^{-z}, \\ f_1(z) &= \alpha_1 e^z + \beta_1 e^{-z}, \\ f_2(z) &= \alpha_2 e^z + \beta_2 e^{-z}, \\ f_3(z) &= \alpha_1\alpha_2 e^z + \beta_1\beta_2 e^{-z}. \end{aligned}$$

Let $\Gamma = \{(a_0 f_0 + a_1 f_1 + a_2 f_2 + a_3 f_3, b_1\eta_1 + b_2\eta_2) \mid a_j, b_k \in \mathbb{Z}\}$. This is given in the Example 3-2.

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