Resolvent equations technique for general variational inclusions

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Abstract: In this paper, we introduce and study a new class of variational inclusions and resolvent equations, respectively, and establish the equivalence between the variational inclusions and the resolvent equations. Using the resolvent equations technique, we construct some new iterative algorithms for solving the classes of varitional inclusions and resolvent equations. Under suitable conditions, the convergence analyses of the iterative algorithms are also studied. Our results include several previously known results as special cases.

Key words: General variational inclusion; general resolvent equations.

1. Introduction. Using the projection technique, Verma [9–11] and Yao [12] established the solvability of the generalized variational inequalities involving the relaxed Lipschitz and relaxed monotone operators. Recently, Noor [4–6], Noor-Noor [7] and Noor-Noor-Rassias [8] introduced and studied some new classes of variational inequalities, variational inclusions, resolvent equations and Wiener-Hopf equations.

Inspired and motivated by the results in [4–12], in this paper, we introduce and study a new class of variational inclusions and resolvent equations, respectively. These classes are more general and include the previously known classes of variational inequalities and variational inclusions and resolvent equations, respectively, as special cases. We establish the equivalence between the variational inclusions and the resolvent equations. Using the resolvent equations technique, we construct some new iterative algorithms for solving the classes of variational inclusions and resolvent equations. Under suitable conditions, the convergence analyses of the iterative algorithms are also studied. The results presented in this paper generalize, improve and unify a number of recent results due to Noor [4, 6], Noor-Noor [7] and Noor-Noor-Rassias [8].

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2. Preliminaries. Throughout this paper, we assume that H is a real Hilbert space endowed with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively, and I denotes the identity operator on H. Let 2^H and $CB(H)$ stand for the families of all nonempty subsets and all nonempty closed bounded subsets of H, respectively. Let $\varphi : H \to R \cup \{+\infty\}$ be a proper convex lower semicontinuous function on H and $\partial\varphi$ denote the subdifferential of function φ. Let $A, B, C \colon H \to 2^H$ be multivalued mappings, $g : H \to H$ be a mapping and $N : H \times H \times H \to H$ be a nonlinear mapping. Suppose that $M : H \to 2^H$ is a multivalued maximal monotone mapping. For each given $f \in H$, we consider the following problem:

Find $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$ such that

$$f \in N(x, y, z) + M(gu). \tag{2.1}$$

Problem (2.1) is called the general variational inclusion.

Definition 2.1 [1]. If M is a maximal monotone from H into 2^H, then for a constant $\rho > 0$, the resolvent operator associate with M is defined by

$$J_M(u) = (I + \rho M)^{-1}(u), \quad \text{for all } \ u \in H.$$

It is known (cf. [1]) that the resolvent operator J_M is single-valued and nonexpansive.

In relation to problem (2.1), we consider the problem of finding $w, u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$ such that

$$N(x, y, z) + \rho^{-1} R_M w = f, \tag{2.2}$$

where $\rho > 0$ is a constant, $R_M = I - J_M$ and J_M is the resolvent operator. The equations of the type (2.2) are called the general resolvent equations. Moreover, if $M(u) = I_K(u)$ is the indicator function of K, the resolvent operator $J_M \equiv P_K$, the projection of H onto K. Consequently, problem (2.2) is equivalent to finding $w, u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$ such that

$$N(x,y,z) + \rho^{-1} Q_K w = f, \tag{2.3}$$

where $Q_K = I - P_K$ and $\rho > 0$ is a constant. The equations (2.3) are called the Wiener-Hopf equations. For the formulations and applications of the resolvent equations and Wiener-Hopf equations, see [4–8].

For a suitable choice of the mappings A, B, C, N, M, the element f, and the space H, one can obtain a number of known and new classes of variational inequalities, variational inclusions, and resolvent equations from the general variational inclusions (2.1) and the general resolvent equations (2.2). Further, these types of variational inclusion and resolvent equations enable us to study many important problems arising in mathematical, regional, physical and engineering sciences in a general and unified framework.

Definition 2.2. A multivalued mapping $A : H \to 2^H$ is said to be strongly monotone with respect to the first argument of $N(\cdot,\cdot,\cdot) : H \times H \times H \to H$, if there exists a constant $r > 0$ such that

$$\begin{aligned}&\langle N(x,\cdot,\cdot) - N(y,\cdot,\cdot), u - v\rangle \\ &\qquad \geq r\|u - v\|^2 \quad \text{for all } x \in Au,\ y \in Av.\end{aligned}$$

Definition 2.3. A multivalued mapping $B : H \to 2^H$ is said to be relaxed Lipschitz with respect to the second argument of $N(\cdot,\cdot,\cdot) : H \times H \times H \to H$, if there exists a constant $r > 0$ such that

$$\begin{aligned}&\langle N(\cdot,x,\cdot) - N(\cdot,y,\cdot), u - v\rangle \\ &\qquad \leq -r\|u - v\|^2 \quad \text{for all } x \in Bu,\ y \in Bv.\end{aligned}$$

Definition 2.4. A multivalued mapping $C : H \to 2^H$ is said to be relaxed monotone with respect to the third argument of $N(\cdot,\cdot,\cdot) : H \times H \times H \to H$, if there exists a constant $r > 0$ such that

$$\begin{aligned}&\langle N(\cdot,\cdot,x) - N(\cdot,\cdot,y), u - v\rangle \\ &\qquad \geq -r\|u - v\|^2 \quad \text{for all } x \in Cu,\ y \in Cv.\end{aligned}$$

Definition 2.5. A mapping $N : H \times H \times H \to H$ is said to be Lipschitz continuous with respect to the first argument if there exists a constant $t > 0$ such that

$$\|N(x,\cdot,\cdot) - N(y,\cdot,\cdot)\| \leq t\|x - y\| \quad \text{for all } x, y \in H.$$

In a similar way, we can define Lipschitz continuity of the mapping $N(\cdot,\cdot,\cdot)$ with respect to the second or third argument.

Definition 2.6. A multivalued mapping $A : H \to CB(H)$ is said to be H-Lipschitz continuous if there exists a constant $r > 0$ such that

$$H(Ax, Ay) \leq r\|x - y\| \quad \text{for all } x, y \in H,$$

where $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(H)$.

Definition 2.7. (i) A mapping $g : H \to H$ is said to be strongly monotone if there exists a constant $\sigma > 0$ such that

$$\langle gx - gy, x - y\rangle \geq \sigma\|x - y\|^2 \quad \text{for all } x, y \in H;$$

(ii) A mapping $g\colon H \to H$ is said to be Lipschitz continuous if there exists a constant $\delta > 0$ such that

$$\|gx - gy\| \leq \delta\|x - y\| \quad \text{for all } x, y \in H.$$

Obviously, if the mapping g is strongly monotone and Lipschitz continuous, then $\delta \geq \sigma$.

3. Iterative algorithms.

Lemma 3.1. *The general variational inclusion* (2.1) *has a solution* $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$ *if and only if the general resolvent equation* (2.2) *has a solution* $w, u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, *where*

$$gu = J_M w, \tag{3.1}$$

$$w = (gu + \rho f - \rho N(x,y,z)) \tag{3.2}$$

and $\rho > 0$ *is a constant.*

Proof. Suppose that the general variational inclusion (2.1) has a solution $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$. Then for $\rho > 0$, we have

$$\begin{aligned}\rho f &\in \rho N(x,y,z) + \rho M(gu) \\ &= -gu + \rho N(x,y,z) + (I + \rho M)(gu),\end{aligned}$$

which implies that

$$gu = J_M(gu + \rho f - \rho N(x,y,z)), \tag{3.3}$$

which ensures that

$$\begin{aligned}&R_M(gu + \rho f - \rho N(x,y,z)) \\ &= gu + \rho f - \rho N(x,y,z) - J_M(gu + \rho f - \rho N(x,y,z)) \\ &= \rho f - \rho N(x,y,z).\end{aligned}$$

That is,

$$N(x,y,z) + \rho^{-1} R_M(w) = f,$$

where $w = gu + \rho f - \rho N(x,y,z)$. Thus (3.1) follows from (3.3).

Conversely, suppose that the general resolvent equation (2.2) has a solution $w, u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$ satisfying (3.1) and (3.2). By virtue of (3.1) and (3.2), we conclude that (3.3) holds. Hence

$$gu+\rho f-\rho N(x,y,z) \in (I+\rho M)(gu) = gu+\rho M(gu),$$

which means that

$$f \in N(x,y,z)+M(gu).$$

This completes the proof. □

Remark 3.1. Lemma 3.1 generalized Theorem 3.1 in [4] and [6], Theorem 5.1 in [7] and Theorem 3.2 in [8].

Using Lemma 3.1 and Nadler's result [3], we can suggest the following iterative algorithms for solving the general resolvent equation (2.2) and the general Wiener-Hopf equation (2.3).

Algorithm 3.1. Let $g\colon H \to H$, $N\colon H\times H\times H \to H$, $A, B, C\colon H \to CB(H)$, $M\colon H \to 2^H$. Let f be a given element in H, $\rho > 0$ be a constant and $g(H) \supseteq J_M(H)$. For given $w_0, u_0 \in H$, $x_0 \in Au_0$, $y_0 \in Bu_0$, $z_0 \in Cu_0$, compute $\{w_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, and $\{z_n\}_{n=0}^{\infty}$ by the iterative schemes

$$gu_n = J_M w_n, \tag{3.4}$$

$$\begin{aligned}
&\|x_n - x_{n+1}\| \\
&\quad \le (1+(n+1)^{-1})H(Au_n, Au_{n+1}),\ x_n \in Au_n,\\
&\|y_n - y_{n+1}\| \\
&\quad \le (1+(n+1)^{-1})H(Bu_n, Bu_{n+1}),\ y_n \in Bu_n,\\
&\|z_n - z_{n+1}\| \\
&\quad \le (1+(n+1)^{-1})H(Cu_n, Cu_{n+1}),\ z_n \in Cu_n,
\end{aligned} \tag{3.5}$$

$$w_{n+1} = (1-\lambda)w_n + \lambda(gu_n + \rho f - \rho N(x_n, y_n, z_n)), \tag{3.6}$$

for all $n \ge 0$, where $\lambda \in (0,1]$ is a parameter.

Algorithm 3.2. Let $g\colon H \to H$, $N\colon H\times H\times H \to H$, $A, B, C\colon H \to CB(H)$. Let f be a given element in H, $\rho > 0$ be a constant and $g(H) \supseteq P_K(H)$. Let K be a closed convex subset of H and φ denote the indicator function on K. For given $w_0, u_0 \in H$, $x_0 \in Au_0$, $y_0 \in Bu_0$, $z_0 \in Cu_0$, compute $\{w_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, and $\{z_n\}_{n=0}^{\infty}$ by the iterative schemes (3.5), (3.6) and $gu_n = P_K w_n$ for all $n \ge 0$.

Remark 3.2. For a suitable choice of mappings N, g, M, A, B, C, element f, closed convex set K and parameter λ, one can obtain a wide class of iterative algorithms for solving various classes of resolvent equations and Wiener-Hopf equations. Algorithm 3.1 and Algorithm 3.2 include a few known algorithms in [4, 6–8] as special cases.

4. Existence and convergence theorems. In this section, we prove two results that deal with the existence of a solution of the general resolvent equations (2.2) and the convergence of an iterative sequence generalized by Algorithm 3.1.

Theorem 4.1. *Let $g : H \to H$ be strongly monotone and Lipschitz continuous with constants σ and δ, respectively, $M : H \to 2^H$ be a maximal monotone mapping with $g(H) \supseteq J_M(H)$. Let N be Lipschitz continuous with respect to the first, second and third arguments with constants β, η and a, respectively, $A, B, C\colon H \to CB(H)$ be H-Lipschitz continuous with constants μ, ξ and b, respectively, and A be strongly monotone with respect to the first argument of N with constant α. Let*

$$\begin{aligned} k &= \sqrt{1-2\sigma+\delta^2},\\ m &= \sqrt{1-2\sigma+\beta^2\mu^2}+\eta\xi+ab,\ \beta\mu \ge \alpha. \end{aligned} \tag{4.1}$$

If there exists a constant $\rho > 0$ such that

$$k+\rho m < 1, \tag{4.2}$$

holds and at least one of the following conditions

$$\begin{aligned}
&m<1,\ |\sigma-(1-k)m| > \sqrt{[\delta^2-(1-k)^2](1-m^2)},\\
&\left|\rho - \frac{\sigma-(1-k)m}{1-m^2}\right| \\
&\quad < \frac{\sqrt{[\sigma-(1-k)m]^2-[\delta^2-(1-k)^2](1-m^2)}}{1-m^2};
\end{aligned} \tag{4.3}$$

$$m=1,\ \sigma+k>1.\ \rho > \frac{\delta^2-(1-k)^2}{2[\sigma+k-1]}; \tag{4.4}$$

$$\begin{aligned}
&m>1,\\
&\left|\rho - \frac{m(1-k)-\sigma}{m^2-1}\right| \\
&\quad > \frac{\sqrt{[\delta^2-(1-k)^2](m^2-1)+[m(1-k)-\sigma]^2}}{m^2-1},
\end{aligned} \tag{4.5}$$

is satisfied, then for each given $f \in H$, there exist $w, u \in H$, $x \in Au$, $y \in Bu$ and $z \in Cu$ satisfying the resolvent equations (2.2) *and* (3.1) *and* (3.2)*, and the sequence $\{w_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ generated by Algorithm* 3.1 *converge, respectively, to w, u, x, y and z strongly in H.*

Proof. In view of Algorithm 3.1, we obtain that

(4.6)
$$\begin{aligned}
&\|w_{n+1}-w_n\|\\
&=\|(1-\lambda)w_n+\lambda(gu_n+\rho f-\rho N(x_n,y_n,z_n))\\
&\quad-(1-\lambda)w_{n-1}\\
&\quad-\lambda(gu_{n-1}+\rho f-\rho N(x_{n-1},y_{n-1},z_{n-1}))\|\\
&\le(1-\lambda)\|w_n-w_{n-1}\|+\lambda\|gu_n-gu_{n-1}\\
&\quad-\rho(N(x_n,y_n,z_n)-N(x_{n-1},y_{n-1},z_{n-1}))\|\\
&\le(1-\lambda)\|w_n-w_{n-1}\|\\
&\quad+\lambda\|\rho(u_n-u_{n-1})-(gu_n-gu_{n-1})\|\\
&\quad+\lambda\rho\|u_n-u_{n-1}\\
&\quad-(N(x_n,y_n,z_n)-N(x_{n-1},y_{n-1},z_{n-1}))\|\\
&\le(1-\lambda)\|w_n-w_{n-1}\|\\
&\quad+\lambda\|\rho(u_n-u_{n-1})-(gu_n-gu_{n-1})\|\\
&\quad+\lambda\rho\|u_n-u_{n-1}\\
&\quad-(N(x_n,y_n,z_n)-N(x_{n-1},y_n,z_n))\|\\
&\quad+\lambda\rho\|N(x_{n-1},y_n,z_n)-N(x_{n-1},y_{n-1},z_n)\|\\
&\quad+\lambda\rho\|N(x_{n-1},y_{n-1},z_n)-N(x_{n-1},y_{n-1},z_{n-1})\|.
\end{aligned}$$

Since g is Lipschitz continuous and strongly monotone, we have

(4.7)
$$\begin{aligned}
&\|\rho(u_n-u_{n-1})-(gu_n-gu_{n-1})\|^2\\
&=\rho^2\|u_n-u_{n-1}\|^2-2\rho\langle gu_n-gu_{n-1},u_n-u_{n-1}\rangle\\
&\qquad+\|gu_n-gu_{n-1}\|^2\\
&\le\rho^2-2\sigma\rho+\delta^2)\|u_n-u_{n-1}\|^2.
\end{aligned}$$

Since A is H-Lipschitz continuous and strongly monotone with respect to the first argument of N, and N is Lipschitz continuous with respect to the first argument, we conclude that

(4.8)
$$\begin{aligned}
&\|u_n-u_{n-1}-(N(x_n,y_n,z_n)-N(x_{n-1},y_n,z_n))\|^2\\
&=\|u_n-u_{n-1}\|^2\\
&\quad-2\langle N(x_n,y_n,z_n)-N(x_{n-1},y_n,z_n),u_n-u_{n-1}\rangle\\
&\quad+\|N(x_n,y_n,z_n)-N(x_{n-1},y_n,z_n)\|^2\\
&\le(1-2\alpha)\|u_n-u_{n-1}\|^2+\beta^2\|x_n-x_{n-1}\|^2\\
&\le(1-2\alpha+\beta^2\mu^2(1+n^{-1})^2)\|u_n-u_{n-1}\|^2.
\end{aligned}$$

Since N is Lipschitz continuous with respect to the second and third arguments, respectively, and B and C are H-Lipschitz continuous, we know that

$$\begin{aligned}
(4.9)\quad&\|N(x_{n-1},y_n,z_n)-N(x_{n-1},y_{n-1},z_n)\|\\
&\qquad\le\xi\eta(1+n^{-1})\|u_n-u_{n-1}\|,
\end{aligned}$$

$$\begin{aligned}
(4.10)\quad&\|N(x_{n-1},y_{n-1},z_n)-N(x_{n-1},y_{n-1},z_{n-1})\|\\
&\qquad\le ab(1+n^{-1})\|u_n-u_{n-1}\|.
\end{aligned}$$

In view of (3.4), (4.1) and (4.7), we infer that

$$\begin{aligned}
&\|u_n-u_{n-1}\|\\
&\le|u_n-u_{n-1}-(gu_n-gu_{n-1})\|\\
&\quad+\|J_Mw_n-J_Mw_{n-1}\|\\
&\le\sqrt{1-2\sigma+\delta^2}\|u_n-u_{n-1}\|+\|w_n-w_{n-1}\|\\
&=k\|u_n-u_{n-1}\|+\|w_n-w_{n-1}\|,
\end{aligned}$$

which implies that

$$(4.11)\qquad \|u_n-u_{n-1}\|\le(1-k)^{-1}\|w_n-w_{n-1}\|.$$

Substituting (4.7)–(4.11) into (4.6), we obtain that

$$(4.12)\qquad \|w_{n+1}-w_n\|\le\theta_n\|w_n-w_{n-1}\|,$$

where

$$\begin{aligned}
\theta_n=(1-\lambda)+\lambda(1-k)^{-1}\big[&\sqrt{\rho^2-2\rho\sigma+\delta^2}\\
&+\rho\sqrt{1-2\alpha+\beta^2\mu^2(1+n^{-1})^2}\\
&+(1+n^{-1})\rho(\xi\eta+ab)\big].
\end{aligned}$$

Put

$$\begin{aligned}
(4.13)\quad\theta=(1-\lambda)+\lambda(1-k)^{-1}\big[&\sqrt{\rho^2-2\rho\sigma+\delta^2}\\
&+\rho\sqrt{1-2\alpha+\beta^2\mu^2}+\rho(\xi\eta+ab)\big].
\end{aligned}$$

It is easy to see that $\theta_n\downarrow\theta$ as $n\to\infty$. It follows from (4.1), (4.2) and (4.13) that

$$\begin{aligned}
(4.14)\quad&\theta<1\Leftrightarrow\sqrt{\rho^2-2\rho\sigma+\delta^2}<1-k-\rho m\\
&\Leftrightarrow(1-m^2)\rho^2-2[\alpha-(1-k)m]\rho<(1-k)^2-\delta^2.
\end{aligned}$$

We now assert that

$$(4.15)\qquad \delta\ge1-k.$$

It follows from (4.1) and (4.2) that $k\in(0,1]$. If $\delta\ge1$, then (4.15) holds; if $\delta\in(0,1)$, we conclude that by virtue of (4.1) and $\delta\ge\sigma$,

$$k^2=1-2\sigma+\delta^2\ge(1-\delta)^2,$$

which implies that $k\ge1-\delta$. That is, (4.15) holds.

It is easy to verify that (4.14), (4.15) and one of (4.3) and (4.4) and (4.5) yield that $\theta<1$. Thus $\theta_n<1$ for n sufficiently large. It is easy to see that (4.12) means that $\{w_n\}_{n=0}^\infty$ is a Cauchy sequence in H. Consequently, there exists $w\in H$ such that $\lim_{n\to\infty}w_n=w$. By virtue of (4.11), we know that the sequence $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence in H, that is, there exists $u \in H$ with $\lim_{n\to\infty} u_n = u$. Note that A, B, C are H-Lipschitz continuous. In view of (3.5), we have

$$\|x_n - x_{n-1}\| \le \mu(1+n^{-1})\|u_n - u_{n-1}\|,$$
$$\|y_n - y_{n-1}\| \le \xi(1+n^{-1})\|u_n - u_{n-1}\|,$$
$$\|z_n - z_{n-1}\| \le b(1+n^{-1})\|u_n - u_{n-1}\|,$$

which imply that $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{z_n\}_{n=0}^{\infty}$ are Cauchy sequences in H. Hence there exists $x, y, z \in H$ such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} z_n = z$. Observe that

$$\begin{aligned} d(x, Au) &= \inf\{\|x - t\| : t \in Au\} \\ &\le \|x_n - x\| + H(Au_n, Au) \\ &\le \|x_n - x\| + \mu\|u_n - u\| \to 0 \end{aligned}$$

as $n \to \infty$. This means that $x \in Au$. Similarly, we have $y \in Bu$, $z \in Cu$. It follows from the continuity of the operators A, B, C, g, J_M, N and (3.4) and (3.6) that

$$gu = J_M w$$

and

$$w = (1-\lambda)w + \lambda(gu + \rho f - \rho N(x, y, z)) \in H,$$

which implies that

$$w = gu + \rho f - \rho N(x, y, z).$$

Lemma 3.1 ensures that $w, u \in H$, $x \in Au$, $y \in Bu$ and $z \in Cu$ is a solution of the resolvent equation (2.2). This completes the proof. □

Remark 4.1. Theorem 4.1 extends, improves and unifies Theorem 4.1 in [4], Theorem 3.2 in [6] and Theorem 5.2 in [7].

Theorem 4.2. *Let g, M, N, A, B, C, P, Q and R be as in Theorem* 4.1. *Assume that B is relaxed Lipschitz with respect to the second argument of N with constant c, and C is relaxed monotone with respect to the third argument of N with constant d. Let*

$$\beta\mu \ge \alpha,\ \eta\xi \ge c,\ k = 2\sqrt{1 - 2\sigma + \delta^2}; \tag{4.16}$$

$$m = \sqrt{1 - 2\alpha + \beta^2\mu^2} + \sqrt{1 - 2c + \eta^2\xi^2} + \sqrt{1 + 2d + a^2b^2}; \tag{4.17}$$

If there exists a constant $\rho > 0$ satisfying (4.2) *and at least one of* (4.3), (4.4) *and* (4.5) *is satisfied, then for each given $f \in H$, there exist $w, u \in H$, $x \in Au$, $y \in Bu$ and $z \in Cu$ satisfying the resolvent equations* (2.2) *and* (3.1) *and* (3.2), *and the sequences $\{w_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, and $\{z_n\}_{n=0}^{\infty}$ generated by Algorithm* 3.1 *converge, respectively, to w, u, x, y and z strongly in H.*

Proof. As in the proof of Theorem 4.1, we have

$$\|u_n - u_{n-1}\| \le (1-k)^{-1}\|w_n - w_{n-1}\|, \tag{4.18}$$

and

(4.19)
$$\begin{aligned} &\|w_{n+1} - w_n\| \\ &\le (1-\lambda)\|w_n - w_{n-1}\| \\ &\quad + \lambda\|gu_n - gu_{n-1} - \rho(u_n - u_{n-1})\| \\ &\quad + \lambda\rho\|u_n - u_{n-1} - (N(x_n, y_n, z_n) \\ &\qquad\qquad - N(x_{n-1}, y_{n-1}, z_{n-1}))\| \\ &\le (1-\lambda)\|w_n - w_{n-1}\| \\ &\quad + \lambda\sqrt{\rho^2 - 2\rho\sigma + \delta^2}\|u_n - u_{n-1}\| \\ &\quad + \lambda\rho\|u_n - u_{n-1} \\ &\qquad\qquad - (N(x_n, y_n, z_n) - N(x_{n-1}, y_n, z_n))\| \\ &\quad + \lambda\rho\|N(x_{n-1}, y_n, z_n) \\ &\qquad\qquad - N(x_{n-1}, y_{n-1}, z_n) + u_n - u_{n-1}\| \\ &\quad + \lambda\rho\|N(x_{n-1}, y_{n-1}, z_n) \\ &\qquad\qquad - N(x_{n-1}, y_{n-1}, z_{n-1}) - (u_n - u_{n-1})\| \\ &\le (1-\lambda)\|w_n - w_{n-1}\| \\ &\quad + \lambda\sqrt{\rho^2 - 2\rho\sigma + \delta^2}\|u_n - u_{n-1}\| \\ &\quad + \lambda\rho\sqrt{1 - 2\alpha + \beta^2\mu^2(1+n^{-1})^2} \\ &\quad + \lambda\rho\|N(x_{n-1}, y_n, z_n) \\ &\qquad\qquad - N(x_{n-1}, y_{n-1}, z_n) + u_n - u_{n-1}\| \\ &\quad + \lambda\rho\|N(x_{n-1}, y_{n-1}, z_n) \\ &\qquad\qquad - N(x_{n-1}, y_{n-1}, z_{n-1}) - (u_n - u_{n-1})\|. \end{aligned}$$

Since B is H-Lipschitz continuous and relaxed Lipschitz with respect to the second argument of N, and C is H-Lipschitz continuous and relaxed monotone with respect to the third argument of N, by (3.5) we conclude that

(4.20)
$$\begin{aligned} &\|N(x_{n-1}, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_n) + u_n - u_{n-1}\|^2 \\ &= \|u_n - u_{n-1}\|^2 \\ &\quad + 2\langle N(x_{n-1}, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_n), u_n - u_{n-1}\rangle \\ &\quad + \|N(x_{n-1}, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_n)\|^2 \\ &\le (1 - 2c + \eta^2\xi^2(1+n^{-1})^2)\|u_n - u_{n-1}\|^2, \end{aligned}$$

and

(4.21)

$$\begin{aligned}
&\|N(x_{n-1}, y_{n-1}, z_n) \\
&\quad - N(x_{n-1}, y_{n-1}, z_{n-1}) - (u_n - u_{n-1})\|^2 \\
&= \|u_n - u_{n-1}\|^2 - 2\langle N(x_{n-1}, y_{n-1}, z_n) \\
&\quad - N(x_{n-1}, y_{n-1}, z_{n-1}), u_n - u_{n-1}\rangle \\
&\quad + \|N(x_{n-1}, y_{n-1}, z_n) - N(x_{n-1}, y_{n-1}, z_{n-1})\|^2 \\
&\leq (1 + 2d + a^2b^2(1 + n^{-1})^2)\|u_n - u_{n-1}\|^2.
\end{aligned}$$

By virtue of (4.18)–(4.21), we know that

$$\begin{aligned}
&\|w_{n+1} - w_n\| \\
&\leq (1 - \lambda)\|w_n - w_{n-1}\| \\
&\quad + \lambda\sqrt{\rho^2 - 2\rho\sigma + \delta^2}\|u_n - u_{n-1}\| \\
&\quad + \lambda\rho\big[\sqrt{1 - 2\alpha + \beta^2\mu^2(1 + n^{-1})^2} \\
&\quad + \sqrt{1 - 2c + \eta^2\xi^2(1 + n^{-1})^2} \\
&\quad + \sqrt{1 + 2d + a^2b^2(1 + n^{-1})^2}\big]\|u_n - u_{n-1}\| \\
&\leq \theta_n\|w_n - w_{n-1}\|,
\end{aligned}$$

where

$$\begin{aligned}
\theta_n = \big[1 - \lambda + \lambda(1 - k)^{-1}\big(&\sqrt{\rho^2 - 2\rho\sigma + \delta^2} \\
&+ \rho\sqrt{1 - 2\alpha + \beta^2\mu^2(1 + n^{-1})^2} \\
&+ \rho\sqrt{1 - 2c + \eta^2\xi^2(1 + n^{-1})^2} \\
&+ \rho\sqrt{1 + 2d + a^2b^2(1 + n^{-1})^2}\big)\big].
\end{aligned}$$

Set

$$\theta = 1 - \lambda + \lambda(1 - k)^{-1}(\sqrt{\rho^2 - 2\rho\sigma + \delta^2} + \rho m).$$

Then $\theta_n \downarrow \theta$ as $n \to \infty$. The rest of the proof identical with the proof of Theorem 4.1. This completes the proof. □

Remark 4.2. For appropriate and suitable choices of the mappings g, M, N, A, B, C, and the space H, Theorem 3.1 in [9–11] and Theorem 3.6 in[12] can be obtained as special cases of Theorem 4.2.

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