The first, the second and the fourth Painlevé transcendents are of finite order

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Abstract: We show that every solution of the first Painlevé equation has the finite growth order. The second and the fourth Painlevé equations have the same property.

Key words: Painlevé equations; growth order.

1. Introduction. Consider the first Painlevé equation

$$\text{(I)} \qquad w'' = 6w^2 + z$$

($' = d/dz$). Every solution of (I) is meromorphic in **C** ([2], [3], for [3] see [6]). In this paper, we prove the following:

Theorem 1.1. *Let $w(z)$ be an arbitrary meromorphic solution of* (I). *Then, $T(r, w) = O(r^C)$, where C is some positive number independent of $w(z)$.*

For the notation in the value distribution theory such as $m(r, f)$, $N(r, f)$, $T(r, f)$, $S(r, f)$, the reader may consult [4]. In the proof, Lemma 2.1 and the auxiliary function given by (2.5) play essential roles. The second and the fourth Painlevé transcendents have the same property (Theorem 4.1). We remark that the constant C in Theorem 1.1 can be replaced by $5/2$, which is proved by another method ([5]).

2. Lemmas. In what follows, $w(z)$ denotes an arbitrary meromorphic solution of (I), and

$$\theta = 2^{-4}, \quad D_0 = \{z \mid |z| \geq 5\}.$$

We begin with the following lemma, whose proof is a modification of M. Hukuhara's argument ([3], [6]).

Lemma 2.1. *Suppose that, for $a \in D_0$, $|w(a)| \leq \theta^2 |a|^{1/2}/6$. Then,*
(i) *$w(z)$ is analytic and bounded for $|z - a| < \delta_a$,*
(ii) *$|w(z)| \geq \theta^2 |a|^{1/2}/5$ for $(5/6)\delta_a \leq |z - a| \leq \delta_a$,*
where δ_a is a positive number such that

$$\text{(2.1)} \quad \theta |a|^{-1/4} \min\left\{1, \frac{\theta |a|^{3/4}}{|w'(a)|}\right\} < \delta_a \leq 3\theta |a|^{-1/4}.$$

Proof. In (I), put $z - a = \rho t$, $\rho = a^{-1/4}$, $w(z) = w(a + \rho t) = \theta a^{1/2} W(t)$. Then,

$$\ddot{W}(t) = 6\theta W(t)^2 + \theta^{-1}(1 + \rho^5 t) \quad \left(\dot{} = \frac{d}{dt}\right).$$

Integrating both sides twice, we have

$$\text{(2.2)} \quad W(t) = W(0) + \dot{W}(0)t + \frac{\theta^{-1}t^2}{2} + g(t),$$

where

$$W(0) = \theta^{-1}a^{-1/2}w(a), \quad \dot{W}(0) = \theta^{-1}a^{-3/4}w'(a),$$
$$g(t) = \frac{\theta^{-1}\rho^5 t^3}{6} + 6\theta \int_0^t \int_0^\tau W(s)^2 ds d\tau.$$

(1) *Case* $|\dot{W}(0)| \leq 1$. We put

$$\eta_0 = \sup\{\eta \mid M(\eta) \leq 8\theta\},$$

where $M(\eta) = \max\{|W(t)| \mid |t| \leq \eta\}$. Then, $\eta_0 > 0$, because $|W(0)| = \theta^{-1}|a|^{-1/2}|w(a)| \leq \theta/6$. Suppose that $\eta_0 < 3\theta$. Observing that, for $|t| \leq \eta_0$ and for $|a| \geq 5$,

$$\text{(2.3)} \quad |g(t)| \leq \frac{\theta^{-1}|\rho|^5|t|^3}{6} + 6\theta \int_0^t \int_0^\tau |W(s)|^2 |ds||d\tau| \leq \frac{\theta^{-1}|\rho|^5(3\theta)^3}{6} + \frac{6\theta(8\theta)^2(3\theta)^2}{2} < \frac{\theta}{4},$$

we obtain from (2.2) that

$$\text{(2.4)} \quad |W(t)| \leq |W(0)| + |t| + \frac{\theta^{-1}|t|^2}{2} + \frac{\theta}{4} < 7.92\theta$$

for $|t| \leq \eta_0$, which contradicts the supposition. Hence $\eta_0 \geq 3\theta$, and (2.3) is valid for $|t| \leq 3\theta$. Furthermore, for $2.5\theta \leq |t| \leq 3\theta$,

$$|W(t)| \geq \frac{\theta^{-1}|t|^2}{2} - |W(0)| - |t| - |g(t)| \geq \left(\frac{2.5^2}{2} - \frac{1}{6} - 2.5 - \frac{1}{4}\right)\theta > \frac{\theta}{5}.$$

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Going back to the original variables, we obtain (i) and (ii) with $\delta_a = 3\theta|a|^{-1/4}$.

(2) *Case* $|\dot{W}(0)| = \kappa > 1$. Putting

$$\eta_1 = \sup\{\eta \mid M(\eta) \le 5\theta\}$$

and supposing $\eta_1 < (2/\kappa)\theta$, we obtain $|g(t)| < \theta/24$ and $|W(t)| \le |W(0)| + \kappa|t| + \theta^{-1}|t|^2/2 + \theta/24 < 4.3\theta$ for $|t| \le \eta_1$, instead of (2.3) and (2.4). This implies $\eta_1 \ge (2/\kappa)\theta$, and hence $|g(t)| < \theta/24$ for $|t| \le (2/\kappa)\theta$. Furthermore, for $(0.8/\kappa)\theta \le |t| \le (1.2/\kappa)\theta$,

$$\begin{aligned}|W(t)| &\ge \kappa|t| - \frac{\theta^{-1}|t|^2}{2} - |W(0)| - |g(t)| \\ &\ge \Big(0.8 - \frac{0.8^2}{2} - \frac{1}{6} - \frac{1}{24}\Big)\theta > \frac{\theta}{5}.\end{aligned}$$

Thus we obtain (i) and (ii) with

$$\delta_a = \frac{1.2\theta|a|^{-1/4}}{\kappa} = \frac{1.2\theta|a|^{-1/4}\cdot\theta|a|^{3/4}}{|w'(a)|},$$

which completes the proof. □

Remark. In Lemma 2.1, since $|a| \ge 5$, the property (ii) can be replaced by

(ii′) $|w(z)| \ge \theta^2|z|^{1/2}/5.5$ for $(5/6)\delta_a \le |z-a| \le \delta_a$.

Lemma 2.2. *There exists a curve* $\Gamma_0 : z = \phi(x)$, $0 \le x < +\infty$ *such that*

(1) x *is the length of* Γ_0 *from* $\phi(0)$ *to* $\phi(x)$;

(2) $|\phi(x)|$ *is monotone increasing and* $|\phi(x)| \to +\infty$ *as* $x \to +\infty$;

(3) $|dz| \le (6/\sqrt{11})d|z|$ *along* Γ_0;

(4) $|w(z)| \ge 2^{-11}|z|^{1/2}$ *along* Γ_0.

Proof. Consider the ray R_0 ($\subset \mathbf{R}$): $z \ge 5$. Start from $z = 5$, and proceed along R_0. Suppose that a point $a \in R_0$, $a \ge 5$ satisfies $|w(a)| \le \theta^2|a|^{1/2}/6$ and $|w(z)| > \theta^2|z|^{1/2}/6$ for $5 \le z < a$. Draw the semi-circle C_a: $|z-a| = \delta_a$, $\operatorname{Re} z \ge 0$ (cf. Lemma 2.1) which crosses $\mathbf{R}$ at a_- and a_+ ($a_- < a_+$). Take points a_-^*, a_+^* ($\operatorname{Re} a_-^* < \operatorname{Re} a_+^*$) on the semi-circle C_a^*: $|z-a| = (5/6)\delta_a$, $\operatorname{Re} z \ge 0$ in such a way that the segments $[a_-, a_-^*]$ and $[a_+^*, a_+]$ come in contact with C_a^*. Let $\gamma(a)$ be a curve which consists of the segments $[a_-, a_-^*]$, $[a_+^*, a_+]$ and the arc $(a_-^* a_+^*)^\frown \subset C_a^*$. Replacing the segment $[a_-, a_+]$ by $\gamma(a)$, we get the curve $\Gamma_1 = (R_0 \setminus [a_-, a_+]) \cup \gamma(a)$. By Lemma 2.1 and Remark, and by a geometric consideration, we have, on Γ_1, $|w(z)| \ge \theta^2|z|^{1/2}/6 > 2^{-11}|z|^{1/2}$ and $|dz| \le (6/\sqrt{11})d|z|$ if $\operatorname{Re} z \le a_+$. Start again from $z = a_+$. Suppose that we first meet a point $b \in \Gamma_1$, $b > a_+$ such that $|w(b)| = \theta^2|b|^{1/2}/6$. By the same argument as above, we obtain the curve $\gamma(b)$. Then it crosses Γ_1 at b'_-, b_+ ($\operatorname{Im} b'_- \ge 0$, $b_+ \in \mathbf{R}$, $\operatorname{Re} b'_- < b_+$). Replacing the part of Γ_1 from b'_- to b_+ by that of $\gamma(b)$ from b'_- to b_+, we get the path Γ_2. On it, the inequalities in (3) and (4) are valid, if $\operatorname{Re} z \le b_+$. Repeat this procedure. For every $l \ge 5$, the modification of the part such that $l \le \operatorname{Re} z \le l+1$ can be done by repeating this procedure finitely many times. The reason is stated as follows. If not so, there exists a sequence $\{a(\nu)\}_{\nu=0}^\infty \subset [l, l+1] \subset \mathbf{R}$ satisfying $\sum_{\nu=0}^\infty \delta_{a(\nu)} \le 1$ and $|w(a(\nu))| \le \theta^2(l+1)^{1/2}/6$. Hence, by (2.1), we may choose a subsequence $\{a(\nu_j)\}_{j=0}^\infty$ satisfying $a(\nu_j) \to a_* \in [l, l+1]$, $w(a(\nu_j)) \to w_* \ne \infty$, $w'(a(\nu_j)) \to \infty$, as $j \to \infty$. Then $w(a_*) = w_* \ne \infty$, $w'(a_*) = \infty$, which is a contradiction. Therefore we get the curve Γ_0 with the desired properties. □

Modifying a circle in a similar way, we obtain

Lemma 2.3. *For an arbitrary pole* $z = \sigma$ *of* $w(z)$ *such that* $|\sigma| > 10$, *there exists a closed Jordan curve* J_σ *with the properties*:

(1) $\sigma \in J_\sigma$;

(2) $J_\sigma \subset \{z \mid \ |\sigma| \le |z| \le |\sigma| + 1\}$;

(3) $|dz| \le (6/\sqrt{11})d(|\sigma| \arg z)$ *along* J_σ;

(4) $|w(z)| \ge 2^{-11}|z|^{1/2}$ *along* J_σ.

Consider the auxiliary function

$$\Phi(z) = w'(z)^2 + \frac{w'(z)}{w(z)} - 4w(z)^3 - 2zw(z). \tag{2.5}$$

A straight-forward computation yields

$$\Phi'(z) + \frac{\Phi(z)}{w(z)^2} = -\frac{z}{w(z)} + \frac{w'(z)}{w(z)^3}.$$

Solving this we have

Lemma 2.4. *Let* $\gamma(z_0, z)$ *be a path starting from* z_0 *and ending at* z, *and* $\gamma(z_0, t)$ *the part of* $\gamma(z_0, z)$ *from* z_0 *to* t ($\in \gamma(z_0, z)$). *If* $w(t) \ne 0$ *on* $\gamma(z_0, z)$, *then*

$$\begin{aligned}\Phi(z) = E(z_0, z)^{-1}\Bigg[&\Phi(z_0) - \frac{E(z_0, z)}{2w(z)^2} \\ &+ \frac{1}{2w(z_0)^2} - \int_{\gamma(z_0,z)} \frac{E(z_0, t)}{2w(t)^4}\big(2tw(t)^3 - 1\big)dt\Bigg],\end{aligned}$$

where $E(z_0, t) = \exp(\int_{\gamma(z_0,t)} w(\tau)^{-2} d\tau)$.

For an arbitrary pole $z = \sigma$, $|\sigma| > 10$, consider the disk $U(\sigma) = \{z \mid |z-\sigma| < \eta(\sigma)\}$ with $\eta(\sigma) = \sup\big\{\eta \ \big|\ |w(z)| > 2|z|^{1/2}$ *in* $|z-\sigma| < \eta \ (\le 1)\big\}$. Then

Lemma 2.5. *In* $U(\sigma)$, $|\Phi(z)| \le K_0|z|^\Delta$. *Here* K_0 *is independent of* σ, *and* Δ ($\ge 3/2$) *is independent of* $w(z)$ *and* σ.

Proof. For $z \in \Gamma_0$, denote by $\Gamma_0(z)$ the part of Γ_0 from the starting point c_0 to z. By Lemma 2.2,

$$|E(c_0,z)^{\pm 1}| \le \exp\Big(\int_{\Gamma_0(z)} \frac{|dt|}{|w(t)|^2}\Big) \le \exp\Big(\frac{6\cdot 2^{22}}{\sqrt{11}}\int_{|c_0|}^{|z|}\frac{d|t|}{|t|}\Big) = O(z^{\Delta'}),$$

$\Delta' = 3\cdot 2^{23}/\sqrt{11}$. Hence, by Lemmas 2.2 and 2.4, for $z_\sigma \in \Gamma_0 \cap J_\sigma$, we have $\Phi(z_\sigma) = O(z_\sigma^{2\Delta'+3/2})$. Since $|\sigma| \le |z_\sigma| \le |\sigma|+1$, by Lemma 2.3, $E(z_\sigma, z) = O(1)$ along the curve J_σ ($\ni z_\sigma, z$). Observing that $\Phi(z_\sigma) = O(z_\sigma^{2\Delta'+3/2})$ and using Lemmas 2.3 and 2.4, we have $\Phi(\sigma) = O(\sigma^{2\Delta'+3/2})$. Applying Lemma 2.4 with $\gamma(z_0,z) = [\sigma, z]$, $z \in U(\sigma)$, we derive the conclusion. □

3. Proof of Theorem 1.1. For an arbitrary pole $z=\sigma$ of $w(z)$, by Lemma 2.5, we have $|\Phi(z)| \le K_0|z|^\Delta$ for $z \in U(\sigma)$. Put $w(z) = u(z)^{-2}$, $z = \sigma + \sigma^{-\Delta/6}s$ in (2.5). Then $v(s) = u(\sigma+\sigma^{-\Delta/6}s)$ satisfies

$$\text{(3.1)}\qquad \frac{dv}{ds}(s) = \sigma^{-\Delta/6}\big(1+h(s,v(s))\big), \quad |h(s,v(s))| < \frac{1}{2}, \quad v(0)=0,$$

as long as

$$\text{(3.2)}\qquad \big|(\sigma+\sigma^{-\Delta/6}s)^{\Delta/6}\big||v(s)| < \varepsilon_0,$$

where the branch of $u(z)$ is taken so that $u'(\sigma) = \sigma^{\Delta/6}(dv/ds)(0) = 1$, and $\varepsilon_0 = \varepsilon_0(K_0)$ is a sufficiently small positive constant independent of σ. Note that (3.2) implies $z = \sigma+\sigma^{-\Delta/6}s \in U(\sigma)$. Put $\eta_* = \sup\{\eta \mid \text{(3.2) } \textit{is valid for } |s|<\eta\}$, and suppose that $\eta_* < \varepsilon_0/4$. Then, integrating (3.1), we derive

$$\text{(3.3)}\qquad \frac{|s|}{2} \le |\sigma^{\Delta/6}|\,|v(s)| \le \frac{3|s|}{2} \le \frac{3\varepsilon_0}{8}$$

for $|s| \le \eta_*$ ($< \varepsilon_0/4$), which implies

$$\big|(\sigma+\sigma^{-\Delta/6}s)^{\Delta/6}\big|\,|v(s)| \le |\sigma^{\Delta/6}||v(s)|\Big(1+\frac{1}{M_0}\Big)^{\Delta/6} \le \frac{\varepsilon_0}{2}$$

for $|s| \le \eta_*$ and for $|\sigma| \ge M_0$, where M_0 is sufficiently large. In case $|\sigma| \ge M_0$, this contradicts the definition of η_*, and hence $\eta_* \ge \varepsilon_0/4$. Therefore, if $|\sigma| \ge M_0$, then (3.3) is valid for $|s| < \varepsilon_0/4$, and $w(z)$ is analytic for $0 < |z-\sigma| < (\varepsilon_0/4)|\sigma|^{-\Delta/6}$. By a well-known argument [1, §4.6], the number of the poles of $w(z)$ in the disk $|z| < r$ does not exceed $O(r^{2+\Delta/3})$. Combining this fact with $m(r,w) = S(r,w)$ (cf. [4, Lemma 2.4.2] and [4, Lemma 1.1.1]), we get $T(r,w) = m(r,w)+N(r,w) = O(r^{2+\Delta/3})$, which completes the proof. □

4. The second and the fourth Painlevé equations. The method above is also applicable to the second and the fourth Painlevé equations

$$\text{(II)}\qquad w'' = 2w^3 + zw + \alpha,$$

$$\text{(IV)}\qquad ww'' = \frac{(w')^2}{2} + \frac{3w^4}{2} + 4zw^3 + 2(z^2-\alpha)w^2 + \beta,$$

$\alpha, \beta \in \mathbf{C}$. For (II), instead of (2.5), we consider the auxiliary function

$$\Phi_{\mathrm{II}}(z) = w'(z)^2 + \frac{w'(z)}{w(z)-\theta_1 z^{1/2}} - w(z)^4 - zw(z)^2 - 2\alpha w(z).$$

Putting $z-a = a^{-1/2}t$, $w(z) = \theta_1 a^{1/2}W(t)$, we show a lemma corresponding to Lemma 2.1, and we choose paths analogous to Γ_0 and J_σ on which $|w(z)-\theta_1 z^{1/2}| \ge c_1|z|^{1/2}$. For (IV), we take

$$\Phi_{\mathrm{IV}}(z) = \frac{w'(z)^2}{w(z)} + \frac{4w'(z)}{w(z)-\theta_2 z} - w(z)^3 - 4zw(z)^2 - 4(z^2-\alpha)w(z) + \frac{2\beta}{w(z)}.$$

Using the change of variables $z-a = a^{-1}t$, $w(z) = \theta_2 aW(t)$, we also choose paths corresponding to Γ_0 and J_σ on which $|w(z)-\theta_2 z| \ge c_2|z|$. Here θ_j and c_j ($j=1,2$) are suitably chosen small positive numbers. These auxiliary functions satisfy

$$\Phi'_{\mathrm{II}}(z) + \frac{\Phi_{\mathrm{II}}(z)}{(w(z)-\theta_1 z^{1/2})^2} = \frac{w'(z)}{(w(z)-\theta_1 z^{1/2})^3}\Big(1-\frac{\theta_1^2}{2}+\frac{\theta_1}{2}z^{-1/2}w(z)\Big) - \frac{1}{(w(z)-\theta_1 z^{1/2})^2}\Big(\theta_1^2 zw(z)^2 + \theta_1 z^{3/2}w(z) + \alpha w(z) + \theta_1\alpha z^{1/2}\Big),$$

and

$$\Phi'_{\mathrm{IV}}(z) + \frac{2(w(z)+\theta_2 z)}{(w(z)-\theta_2 z)^2}\Phi_{\mathrm{IV}}(z) = \frac{4w'(z)}{(w(z)-\theta_2 z)^3}\big((2+\theta_2)w(z)+(2-\theta_2)\theta_2 z\big) - \frac{4}{(w(z)-\theta_2 z)^2}\Big(\theta_2^2 z^2 w(z)^2 + 2\theta_2 z^2 w(z) \times (w(z)+\theta_2 z) + 4\theta_2(z^2-\alpha)zw(z) - 2\beta\Big),$$

respectively. By an argument analogous to that in Section 3, we have

Theorem 4.1. *Let $w(z)$ be an arbitrary meromorphic solution of* (II) (*resp.* (IV)). *Then $T(r,w) = O(r^{C'})$ (resp. $O(r^{C''})$), where C' (resp. C'') is some positive number independent of $w(z)$.*

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