On boundedness of a function on a Zalcman domain

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Abstract: We consider boundedness of a funciton defined by an infinite product which is used to study a uniqueness theorem on a plane domain and the point separation problem of a two-sheeted covering Riemann surface. We show that there is such an infinite product that it converges but the function defined by it is not bounded on arbitrary Zalcman domain.

Key words: Bounded analytic function; Zalcman domain; uniqueness theorem.

1. Introduction. Let $\Delta_0 = \{z \in \mathbf{C} : 0 < |z| < 1\}$ be a punctured disc and $\{c_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence with $0 < c_n < 1$ satisfying that $\lim_{n\to\infty} c_n = 0$. Let Δ_n $\bar{\Delta}(c_n, r_n)$ be mutually disjoint closed dises contained in Δ_0 centered at c_n with radii r_n. The condition that Δ_n 's are mutually disjoint is equivalent to the following:

$$c_{n+1} + r_{n+1} < c_n - r_n \tag{1}$$

Set $R := R(c_n, r_n) := \Delta_0 \setminus \cup_{n=1}^{\infty} \Delta_n$. We call a domain of this form a a *Zalcmandomain*. We say that the *uniquenesstheorem* is valid for $H^\infty(R)$ at $z = 0$ if the following condition is fulfulled: if $f \in H^\infty(R)$ satisfice

$$\lim_{z<0, z\to 0} f^{(k)}(z) = 0 \quad (k = 0, 1, 2, \dots)$$

then f vanishes identically on R. For an unlimited smooth two-sheeted covering surface $(\widetilde{R}, R, \pi)$ or R with the projection map π, we say that the *Myrbcrg phenomenon* occurs if we have $H^\infty(\widetilde{R}) = H^\infty(R)\circ\pi$. The Validity of the uniqueness theorem implies the occurance of the Myrberg phenomenon. (See [1].) In this short note we are conccrned with boundedness and unboundedness of the following infinite product:

$$p(z) := \prod_{n=1}^{\infty} \frac{z}{z - c_n} = \prod_{n=1}^{\infty} \left(1 + \frac{c_n}{z - c_n}\right).$$

When $|c_n| < |z|/2$, we have

$$\frac{c_n}{2|z|} \le \left|\frac{c_n}{z - c_n}\right| \le \frac{2c_n}{|z|}.$$

The product $p(z)$ converges and defines a meromorphic function on $\bar{\mathbf{C}}\setminus\{0\}$ if and only if the sequence $\{c_n\}_{n=1}^{\infty}$ satisfies

$$\sum_{n=1}^{\infty} a_n < \infty. \tag{2}$$

In partieular, $p(z)$ is holomorphic on $\bar{R}\setminus\{0\}$ as far as (2) is satisfied. As we describe in the next section, $p(z)$ relates to the uniqueness theorem. So, it is interesting problem to study how small r_n's can be chosen depending on a given sequence $\{c_n\}_{n=1}^{\infty}$ in order that $p(z)$ is boundede on $R(c_n, r_n)$. THe aim of this note is to show that there exists a sequence $\{c_n\}_{n=1}^{\infty}$ such that $p(z)$ is not bounded on $R(c_n, r_n)$ for any $\{r_n\}_{n=1}^{\infty}$.

2. Some related results. Here we list some related results on the function $p(z)$. While some of these results were proved under a stronger condition than (2), the same proofs go through under the condition (2) and we omit the proofs.

Theorem A ([2])**.**

$$\lim_{z<0, z\to 0} p^{(k)}(z) = 0 \quad (k = 0, 1, 2, \dots)$$

Corollary. *If $p(z)$ is bounded on R, then the uniqueness theorem does not hold for $H^\infty(R)$ at $z =$ 0.*

Since $p(z)$ is holomorphic on $\bar{R}\setminus\{0\}$,

$$M_n := \sup_{z\in\partial\Delta_n} |p(z)|$$

is finite for each $n \in \mathbf{N}$. Moreover, we can see

$$\sup_{z\in R} |p(z)| = \sup_{n\in\mathbf{N}} M_n.$$

(See [1].)

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Lemma B. *Suppose that sequences* $\{c_n\}_{n=1}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *satisfy* (1) *and* (2). *Then,*

$$\frac{c_n}{r_n}\prod_{m=1}^{n-1}\frac{c_n}{c_m-c_n}\prod_{m=n+1}^{\infty}\frac{c_n}{c_n-c_m}\le M_n$$
$$\le\frac{c_n+r_n}{r_n}\prod_{m=1}^{n-1}\frac{c_n+r_n}{c_m-(c_n+r_n)}$$
$$\times\prod_{m=n+1}^{\infty}\frac{c_n-r_n}{(c_n-r_n)-c_m}.$$

In this lemma, $\{c_n\}_{n=1}^{\infty}$ was only assumed to satisfy (2). With an additional condition on $\{c_n\}_{n=1}^{\infty}$, the following neccesary and sufficient condition in order that $p(z)\in H^{\infty}(R)$ was given in [2].

Theorem C. *Suppose that a sequence* $\{c_n\}_{n=1}^{\infty}$ *satisfies*

$$\limsup_{n\to\infty}\frac{c_{n+1}}{c_n}<1. \tag{3}$$

Then,

$$p\in H^{\infty}(R)\Longleftrightarrow\sup_{n\in\mathbf{N}}\frac{c_n^n}{c_1\cdots c_{n-1}r_n}<\infty. \tag{4}$$

For instance, let $c_n=2^{-n}$. With respect to the radii $\{r_n\}_{n=1}^{\infty}$, we define $\{N(n)\}_{n=1}^{\infty}$ by the relation that $r_n=2^{-nN(n)}$. Then, (4) is written also in the following form:

$$\begin{aligned}&p(z)\in H^{\infty}(R(2^{-n},2^{-nN(n)}))\\&\Longleftrightarrow\sup_{n\in\mathbf{N}}n\left(N(n)-\frac{n+1}{2}\right)<\infty\end{aligned} \tag{5}$$

(also cf. [1]). Set $r_n=2^{-n(n+1)/2}$. Then, the two sequences $\{c_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ satisgy (1). And by (5), we see that

$$p(z)\in H^{\infty}\left(R\left(2^{-n},2^{-n(n+1)/2}\right)\right).$$

3. Unboundedness of p on arbitrary Zalcman domain. Now we show that there exists a sequence $\{c_n\}_{n=1}^{\infty}$ such that the function p is not bounded on $R(c_n,r_n)$ for any $\{r_n\}_{n=1}^{\infty}$.

Theorem 1. *Let* $c_n=1/n^2$. *Then, the function* $p(z)$ *is no longer boundede on* $R(c_n,r_n)$ *for any choice of* $\{r_n\}_{n=1}^{\infty}$ *satisfying* (1). (*Note that the condition* (3) *does not hold for this* $\{c_n\}_{n=1}^{\infty}$.)

Proof. Suppose that $p(z)$ is in $H^{\infty}(R(1/n^2,r_n))$ for some $\{r_n\}_{n=1}^{\infty}$. Since $\sum_{n=1}^{\infty}1/n^2<\infty$, and the assumption in the Lemma B is fulfilled, we have that

$$\begin{aligned}M_n&\ge\frac{c_n}{r_n}\prod_{m=1}^{n-1}\frac{c_n}{c_m-c_n}\prod_{m=n+1}^{\infty}\frac{c_n}{c_n-c_m}\\&=\frac{1}{n^2r_n}\prod_{m=1}^{n-1}\frac{n^{-2}}{m^{-2}-n^{-2}}\prod_{m=n+1}^{\infty}\frac{n^{-2}}{n^{-2}-m^{-2}}\\&=\frac{1}{n^2r_n}\prod_{m=1}^{n-1}\frac{m^2}{n^2-m^2}\prod_{m=n+1}^{\infty}\frac{m^2}{m^2-n^2}.\end{aligned} \tag{6}$$

The right hand side of (6) is calcurated as follows. For the first product, we have that

$$\begin{aligned}\prod_{m=1}^{n-1}\frac{m^2}{n^2-m^2}&=\prod_{m=1}^{n-1}\frac{m^2}{(n-m)(n+m)}\\&=\frac{\{(n-1)!\}^2}{(n-1)!(n+1)\cdots(2n-1)}\\&=\frac{(n-1)!}{(n+1)\cdots(2n-1)}.\end{aligned} \tag{7}$$

Next, for $k>n+1$, we have that

$$\begin{aligned}&\prod_{m=n+1}^{k}\frac{m^2}{m^2-n^2}=\prod_{m=n+1}^{k}\frac{m^2}{(m-n)(m+n)}\\&=\frac{(n+1)^2(n+2)^2\cdots k^2}{1\cdot2\cdots(k-n)\cdot(2n+1)(2n+2)\cdots(k+n)}\\&=\frac{(2n)!(k!)^2}{(k-n)!(k+n)!(n!)^2}.\end{aligned}$$

Since

$$\begin{aligned}&\lim_{k\to\infty}\frac{(k!)^2}{(k-n)!(k+n)!}\\&=\lim_{k\to\infty}\frac{(k-n+1)(k-n+2)\cdots k}{(k+1)(k+2)\cdots(k+n)}=1,\end{aligned}$$

we have that

$$\begin{aligned}&\prod_{m=n+1}^{\infty}\frac{m^2}{m^2-n^2}\\&=\lim_{k\to\infty}\prod_{m=n+1}^{k}\frac{m^2}{m^2-n^2}-\frac{(2n)!}{(n!)^2}.\end{aligned} \tag{8}$$

From (7) and (8), it follows that

$$\begin{aligned}&\prod_{m=1}^{n-1}\frac{m^2}{n^2-m^2}\prod_{m=n+1}^{\infty}\frac{m^2}{m^2-n^2}\\&=\frac{(n-1)!}{(n+1)\cdots(2n-1)}\frac{(2n)!}{(n!)^2}\\&=\frac{(2n)!(n-1)!}{(2n-1)!n!}=2.\end{aligned} \tag{9}$$

By (6) and (9), we see that

$$\begin{aligned} &M_n \\ &\geq \frac{1}{n^2 r_n} \prod_{m=1}^{n-1} \frac{m^2}{n^2 - m^2} \prod_{m=n+1}^{\infty} \frac{m^2}{m^2 - n^2} \\ &= \frac{2}{n^2 r_n}. \end{aligned}$$

Since the point $1/(n+1)^2$ is not in Δ_n, we have that $1/(n+1)^2 < 1/n^2 - r_n$. This implies that $n^2 r_n < 3/n$. Therefore,

$$\sup_{n\in\mathbf{N}} M_n \geq \sup_{n\in\mathbf{N}} \frac{2}{n^2 r_n} > \sup_{n\in\mathbf{N}} \frac{2n}{3} = \infty.$$

This contradiets the assumption that the function $p(z)$ is boundede on $R(1/n^2, r_n)$ for some $\{r_n\}_{n=1}^{\infty}$. □

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