Uniqueness in the inverse scattering problem for Hartree type equation

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Abstract: In this paper we consider the inverse scattering problem for the following Hartree type equation:

$$i\frac{\partial u}{\partial t} = -\Delta u + (|x|^{-\sigma} * |u|^2)u.$$

We prove the uniqueness theorem on the inverse scattering problem with respect to the power σ.

Key words: Inverse scattering; uniqueness theorem.

1. Introduction. In this paper we consider the following Hartree type equation:

$$(1.1)\quad i\frac{\partial u}{\partial t} = -\Delta u + F(u), \quad (t,x) \in \mathbf{R} \times \mathbf{R}^n,$$

where Δ is the n-dimensional Laplacian in x, and

$$(1.2)\quad F(u) = (|x|^{-\sigma} * |u|^2)u = \left(\int_{\mathbf{R}^n} |x-y|^{-\sigma}|u(t,y)|^2 dy\right) u(t,x).$$

The inverse scattering problem for the nonlinear Schrödinger equation has been studied by Strauss [7], Weder ([9], [10], [12]) and Watanabe [8]. In [7] (pp. 64–67), the power nonlinearity case;

$$F(u) = V(x)|u|^{p-1}u$$

was studied. Weder ([9], [12]) studied the nonlinear Schrödinger equation with a potential;

$$F(u) = V_0(x)u + \lambda|u|^{p-1}u, \quad \lambda \in \mathbf{R}^n$$

and

$$F(u) = V_0(x)u + \sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)}u.$$

A cubic convolution nonlinearity case;

$$F(u) = q(x)u + \mu(|x|^{-\sigma} * |u|^2)u, \quad \mu \in \mathbf{R}^n$$

was studied in Watanabe [8]. In those papers, it was shown that if one of powers (p, j_0, j, or σ in each cases) is given in some suitable conditions, then coefficients ($V(x), V_0(x), \lambda, V_j(x), q(x), \mu$) are uniquely determined from the scattering operator. In addition, a method for the reconstruction of coefficients were given.

Then the following problem arises.

Problem. *When the power* (p *or* σ) *is unknown, can we determine the power from the scattering operator?*

As far as the author knows, there are no results on this problem. In this paper we shall obtain the uniqueness theorem on this inverse scattering problem for Hartree type equation (1.1), (1.2).

Before stating our theorem, we give notations and introduce a result on the scattering problem for the equation (1.1), (1.2).

Notation and function spaces. For a Banach space Z, $L^p(Z) = L^p(\mathbf{R}; Z)$ is the space of all Z valued L^p functions in $\mathbf{R}^n$. The function space $\mathcal{S}$ is indefinitely differentiable on $\mathbf{R}^n$ and all of whose derivatives remain bounded when multiplied by polynominals. Let $\mathcal{F}\phi$ or $\hat{}$ be the Fourier transform of ϕ defined by

$$\mathcal{F}\phi(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{-ix\xi}\phi(x)\, dx.$$

The inverse Fourier transform $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1}\phi(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{ix\xi}\phi(\xi)\, d\xi.$$

For $r \in \mathbf{R}$ and $1 \le p \le \infty$, let $H^{r,p} = H^{r,p}(\mathbf{R}^n)$ be the completions of C_0^∞ with respect to the norm,

$$\|f\|_{H^{r,p}} = \|\mathcal{F}^{-1}\{\langle\xi\rangle^r \hat{f}(\xi)\}\|_{L^p}.$$

Different positive constants might be denoted by

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the same letter C.

Put

$$W = L^3(\mathbf{R}; H^{1,q}) \cap L^\infty(\mathbf{R}; H^{1,2}),$$

where q satisfies

$$\frac{1}{q} = \frac{1}{2} - \frac{2}{3n}. \tag{1.3}$$

The following result on the scattering problem for equation (1.1), (1.2) has been known.

Theorem (Mochizuki [3]). *If σ satisfies $2 \le \sigma \le 4$ and $\sigma < n$, then there exists $\rho > 0$ with the following properties: If $\phi_- \in \mathcal{H}_\rho := \{\phi \in H^{1,2} : \|\phi\|_{H^{1,2}} < \rho\}$, then there exists a unique solution $u \in W$ of (1.1), (1.2) such that*

$$\|u(t) - e^{-itH_0}\phi_-\|_{H^{1,2}} \to 0 \quad \text{as } t \to -\infty.$$

Furthermore, there exists a unique $\phi_+ \in H^{1,2}$ such that

$$\|u(t) - e^{-itH_0}\phi_+\|_{H^{1,2}} \to 0 \quad \text{as } t \to \infty.$$

The map $S : \phi_- \to \phi_+$, which is called a scattering operator, is defined on a neighborhood of 0 in $H^{1,2}$ and represented as

$$S\phi(x) = \phi(x) + \int_{-\infty}^{\infty} e^{itH_0} F(u)\, dt, \tag{1.4}$$

where $H_0 = -\Delta$ and u is a solution of (1.1), (1.2) with the initial data $\phi \in \mathcal{H}_\rho$ at $t = -\infty$.

Now we state our theorem. Put

$$S_j\phi(x) = \phi(x) + \int_{-\infty}^{\infty} e^{itH_0}(|x|^{-\sigma_j} * |u_j(t,x)|^2) \times u_j(t,x)\, dt, \quad j = 1, 2,$$

where $u_j(x)$ is a solution of (1.1), (1.2) with σ_j instead of σ.

Theorem 1.1. *Assume that the power σ satisfies $2 \le \sigma \le 4$ and $\sigma < n$. If $S_1 = S_2$, then $\sigma_1 = \sigma_2$.*

Remark 1.1. In Hayashi-Tsutsumi [1], the scattering problem for equation (1.1), (1.2) was studied under the condition $1 < \sigma < \min(4, n)$. Using their results, we can also obtain Theorem 1.1 for $1 < \sigma < \min(4, n)$ in the same way as we shall prove.

This paper is organized as follows:
In Section 2 we give some preliminary results used throughout this paper. Theorem 1.1 is proved in Section 3.

2. Preliminaries. We summarize some useful lemmas in this section.

Lemma 2.1. *Let $0 < \sigma < n$. Then for any f, $g \in \mathcal{S}$,*

$$(|x|^{-\sigma} * f, g) = \pi^{n/2}\gamma(\sigma)\Big(\Big|\frac{\xi}{2}\Big|^{-n+\sigma}\hat{f}, \hat{g}\Big),$$

where

$$\gamma(\sigma) = \frac{\Gamma\big((n-\sigma)/2\big)}{\Gamma(\sigma/2)}$$

and Γ is the Gamma function.

For Lemma 2.1, see, e.g., Stein [4].

Lemma 2.2. *Let $2 \le \sigma \le 4$ and $\sigma < n$, $1/q = 1/2 - 2/(3n)$. Then there exist positive constants C such that*

$$\|e^{-itH_0}\phi\|_W \le C\|\phi\|_{H^{1,2}}, \quad \text{for } \phi \in H^{1,2},$$
$$\|(|x|^{-\sigma} * [fg])h\|_{H^{1,2}} \le C\|f\|_{H^{1,q}}\|g\|_{H^{1,q}}\|h\|_{H^{1,q}} \quad \text{for } f, g, h \in H^{1,q},$$
$$\Big\|\int_{-\infty}^{t} e^{-i(t-\tau)H_0}(|x|^{-\sigma} * |u(\tau,x)|^2)u(\tau,x)\, d\tau\Big\|_W \le C\|u\|_W^3, \quad \text{for } u \in W.$$

Lemma 2.3. *Let $2 \le \sigma \le 4$ and $\sigma < n$. Then for a solution $u(t,x)$ of the integral equation*

$$u(t,x) = e^{-itH_0}(\varepsilon\phi) + \int_{-\infty}^{t} e^{-i(t-\tau)H_0}(|x|^{-\sigma} * |u(\tau,x)|^2)u(\tau,x)\, d\tau,$$

we have

$$\|u\|_W \le C\varepsilon\|\phi\|_{H^{1,2}} \quad \text{for } \phi \in H^{1,2}.$$

For Lemma 2.2 and Lemma 2.3, see, e.g., Mochizuki [3].

Lemma 2.4. *Let $2 \le \sigma \le 4$ and $\sigma < n$. Then for any $\phi \in H^{1,2}$,*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^3}\big((S - I)(\varepsilon\phi), \phi\big) = \int_{\mathbf{R}}\int_{\mathbf{R}^n} (|x|^{-\sigma} * |e^{-itH_0}\phi|^2)|e^{-itH_0}\phi|^2\, dx\, dt, \tag{2.1}$$

where S is (1.4).

Proof. Let $v = e^{-itH_0}(\varepsilon\phi)$ and $w = u - v$. Then it follows that

$$\begin{aligned}
&\big((S - I)(\varepsilon\phi), \phi\big) \\
&= \int_{\mathbf{R}} \big((|x|^{-\sigma} * |u|^2)u, e^{-itH_0}\phi\big)\, dt \\
&= I_1 + I_2 + I_3 \\
&\quad + \varepsilon^3 \int_{\mathbf{R}}\int_{\mathbf{R}^n} (|x|^{-\sigma} * |e^{-itH_0}\phi|^2)|e^{-itH_0}\phi|^2\, dx\, dt,
\end{aligned}$$

where

$$I_1 = \int_{\mathbf{R}} \big((|x|^{-\sigma} * |u|^2)w, e^{-itH_0}\phi\big)\,dt,$$
$$I_2 = \int_{\mathbf{R}} \big((|x|^{-\sigma} * [u\bar{w}])v, e^{-itH_0}\phi\big)\,dt,$$
$$I_3 = \int_{\mathbf{R}} \big((|x|^{-\sigma} * [w\bar{v}])v, e^{-itH_0}\phi\big)\,dt.$$

By Schwarz inequarity, Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}
&|I_2| \\
&\le \int_{\mathbf{R}} \big|\big((|x|^{-\sigma} * [u\bar{w}])v, e^{-itH_0}\phi\big)\big|\,dt \\
&\le C\int_{\mathbf{R}} \|\,(|x|^{-\sigma} * [u\bar{w}])v\|_{H^{1,2}}\,dt \\
&\le C\int_{\mathbf{R}} \|u\|_{H^{1,q}}\|w\|_{H^{1,q}}\|v\|_{H^{1,q}}\,dt \\
&\le C\int_{\mathbf{R}} \|w\|_{H^{1,q}}(\|u\|_{H^{1,q}}^2 + \|v\|_{H^{1,q}}^2)\,dt \\
&\le C\left(\int_{\mathbf{R}} \|w\|_{H^{1,q}}^3\,dt\right)^{1/3} \\
&\quad \times\left\{\left(\int_{\mathbf{R}} \|u\|_{H^{1,q}}^3\,dt\right)^{2/3} + \left(\int_{\mathbf{R}} \|v\|_{H^{1,q}}^3\,dt\right)^{2/3}\right\} \\
&\le C\left\|\int_{-\infty}^{t} e^{-i(t-\tau)H_0}(|x|^{-\sigma} * |u(\tau,x)|^2)u(\tau,x)\,d\tau\right\|_W \\
&\quad \times(\|u\|_W^2 + \|v\|_W^2) \\
&\le C\|u\|_W^3(\|u\|_W^2 + \varepsilon^2\|\phi\|_{H^{1,2}}^2) \\
&\le C\varepsilon^5\|\phi\|_{H^{1,2}}^5.
\end{aligned}$$

In the same way we have $|I_1| \le C\varepsilon^5\|\phi\|_{H^{1,2}}^5$, $|I_3| \le C\varepsilon^5\|\phi\|_{H^{1,2}}^5$. Hence we get

$$\frac{1}{\varepsilon^3}\big((S-I)(\varepsilon\phi),\phi\big) \longrightarrow \int_{\mathbf{R}}\int_{\mathbf{R}^n} (|x|^{-\sigma} * |e^{-itH_0}\phi|^2)|e^{-itH_0}\phi|^2\,dx\,dt,$$

as $\varepsilon \to 0$. □

3. Proof of Theorem 1.1.

Put

$$I_j[\phi] = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\big((S_j - I)(\varepsilon\phi),\phi\big), \quad j = 1, 2.$$

From Lemma 2.1 and Lemma 2.4

$$\begin{aligned}
I_j[\phi] &= \int_{\mathbf{R}}\int_{\mathbf{R}^n} (|x|^{-\sigma_j} * |e^{-itH_0}\phi|^2)|e^{-itH_0}\phi|^2\,dx\,dt \\
&= C\int_{\mathbf{R}}\int_{\mathbf{R}^n} \frac{\gamma(\sigma_j)}{|\xi/2|^{n-\sigma_j}}\big|\mathcal{F}(|e^{-itH_0}\phi|^2)(\xi)\big|^2\,d\xi\,dt \\
&= C\int_{\mathbf{R}}\int_{\mathbf{R}^n} \frac{\gamma(\sigma_j)}{|\xi|^{n-\sigma_j}}\big|\mathcal{F}(|e^{-itH_0}\phi|^2)(2\xi)\big|^2\,d\xi\,dt.
\end{aligned}$$

Subtracting I_2 from I_1, by the assumption $S_1 = S_2$ we obtain for any $\phi \in H^{1,2}$

$$(3.1)\quad 0 = \int_{\mathbf{R}}\int_{\mathbf{R}^n}\left(\frac{\gamma(\sigma_1)}{|\xi|^{n-\sigma_1}} - \frac{\gamma(\sigma_2)}{|\xi|^{n-\sigma_2}}\right) \times |\mathcal{F}(|e^{-itH_0}\phi|^2)(2\xi)|^2\,d\xi\,dt.$$

We assume that $\sigma_1 > \sigma_2$. If $|\xi|$ is sufficiently small, then we have

$$\frac{\gamma(\sigma_1)}{|\xi|^{n-\sigma_1}} - \frac{\gamma(\sigma_2)}{|\xi|^{n-\sigma_2}} < 0.$$

Taking $\psi \in \mathcal{S}$ such that $\operatorname{supp}\hat{\psi} \subset \{\xi : |\xi| \le \varepsilon',\ \varepsilon'$ is sufficiently small$\}$, then

$$\operatorname{supp}\left\{\big|\mathcal{F}(|e^{-itH_0}\psi|^2)(2\xi)\big|^2\right\} \subset \{\xi : |\xi| \le \varepsilon'\}.$$

Moreover, $\big|\mathcal{F}(|e^{-itH_0}\psi|^2)(2\xi)\big|^2 > 0$ on a neighborhood of $\xi = 0$. Hence for such ψ we have

$$\int_{\mathbf{R}}\int_{\mathbf{R}^n}\left(\frac{\gamma(\sigma_1)}{|\xi|^{n-\sigma_1}} - \frac{\gamma(\sigma_2)}{|\xi|^{n-\sigma_2}}\right) \times \big|\mathcal{F}(|e^{-itH_0}\psi|^2)(2\xi)\big|^2\,d\xi\,dt < 0.$$

This contradicts with (3.1). Therefore we have completed the proof of Theorem 1.1. □

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