On the defining relations of the simply-laced elliptic Lie algebras

By Tadayoshi TAKEBAYASHI

College of Industrial Technology, Nihon University, 1-2-1, Izumicho, Narashino, Chiba 275-8575

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Abstract: We rewrite the defining relations [5] of the simply-laced elliptic Lie algebras in terms of the extended elliptic Cartan matrix by considering the extended elliptic diagram.

Key words: Elliptic root system; elliptic Lie algebra; elliptic Cartan matrix.

1. Introduction. K. Saito and D. Yoshii [5] introduced the simply-laced elliptic Lie algebra $\tilde{\mathfrak{g}}(R)$ for the simply-laced elliptic root system R ([4]), whose derived algebra $\mathfrak{g}(R) := [\tilde{\mathfrak{g}}(R), \tilde{\mathfrak{g}}(R)]$ is isomorphic to 2-toroidal Lie algebra [3] which is the universal central extension of the tensor of a Lie algebra with the Laurent series of two variables. According to the work of Borchards [2], they consider a Lie algebra V_Q/DV_Q as a quotient of the vertex algebra V_Q attached to an even lattice Q, and constructed the elliptic Lie algebra $\tilde{\mathfrak{g}}(R)$ as a subalgebra of V_Q/DV_Q. If R is a simply-laced finite or affine root system, then $\mathfrak{g}(R)$ is isomorphic to a finite or affine Kac-Moody algebra [1], respectively. The defining relations of the generators of $\tilde{\mathfrak{g}}(R)$ in terms of the elliptic diagram have been described in [5]. In this article, we rewrite the defining relations more simply by considering the extended elliptic diagram consisting of all pairs of α_i, α_i^* $(0 \le i \le l)$ for the sake of explicitness, although the results are already intrinsically in [5].

2. Simply-laced elliptic Lie algebras. We recall the elliptic Lie algebra $\tilde{\mathfrak{g}}(R)$ and its defining relations. Let $\Gamma_{\text{ell}} = \Gamma(R, G)$ be the elliptic diagram of a simply-laced marked elliptic root system (R, G) ([4], [5]). Let $Q(R)$ be the root lattice and $F_Q := \mathbf{Q} \otimes_{\mathbf{Z}} Q(R)$. Let $(\widetilde{F_{\mathbf{Q}}}, \tilde{I})$ be its non degenerate hull and $\tilde{\mathfrak{h}} := \mathrm{Hom}_{\mathbf{Q}}(\widetilde{F_{\mathbf{Q}}}, \mathbf{Q})$. Explicitly, $R = R_f + \mathbf{Z}b + \mathbf{Z}a$, $Q(R) = Q_f \oplus \mathbf{Z}b \oplus \mathbf{Z}a$, $\widetilde{F_{\mathbf{Q}}} = F_{\mathbf{Q}} \oplus \mathbf{Q}\Lambda_b \oplus \mathbf{Q}\Lambda_a$, and $\tilde{I}(\Lambda_a, a) = \tilde{I}(\Lambda_b, b) = 1$, $\tilde{I}(\Lambda_a, b) = \tilde{I}(\Lambda_b, a) = 0$, $\tilde{I}(\Lambda_a, \Gamma_f) = \tilde{I}(\Lambda_b, \Gamma_f) = 0$, where R_f, Q_f and Γ_f are the finite root, root lattice and Dynkin diagram, respectively. Further, $\tilde{\mathfrak{h}} = \mathfrak{h}_f \oplus \mathbf{Q}h_{a^\vee} \oplus \mathbf{Q}h_{b^\vee} \oplus \mathbf{Q}h_{\Lambda_a} \oplus \mathbf{Q}h_{\Lambda_b} = \bigoplus_{\alpha \in \Gamma_{\text{ell}}} \mathbf{Q}h_{\alpha^\vee} \oplus \mathbf{Q}h_{\Lambda_a} \oplus \mathbf{Q}h_{\Lambda_b}$, $\mathfrak{h}_f := \bigoplus_{\alpha \in \Gamma_f} \mathbf{Q}h_{\alpha^\vee}$, $\alpha^\vee := 2\alpha/\{I(\alpha, \alpha)\}$ for $\alpha \in \Gamma_{\text{ell}}$, with the inner product $\langle h_x, y \rangle := \tilde{I}(x, y)$ for x, $y \in \widetilde{F_{\mathbf{Q}}}$.

Definition 2.1 (K. Saito and D. Yoshii [5])**.** The elliptic Lie algebra $\tilde{\mathfrak{g}}(R)$ is the algebra generated by the following generators and relations.

generators: $\tilde{\mathfrak{h}}$ and $\{E^\alpha \mid \alpha \in \pm\Gamma_{\text{ell}}\}$

relations:

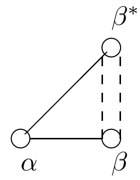
0. $\tilde{\mathfrak{h}}$ is abelian

I. $[h, E^\alpha] = \langle h, \alpha \rangle E^\alpha$

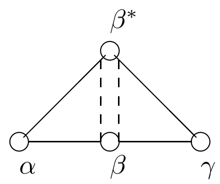
II.1. $[E^\alpha, E^{-\alpha}] = -h_{\alpha^\vee}$
$[E^\alpha, E^\beta] = 0$ for $I(\alpha, \beta) \ge 0$

II.2. $(adE^\alpha)^{1-\langle h_{\alpha^\vee}, \beta \rangle} E^\beta = 0$ for $I(\alpha, \beta) \le 0$

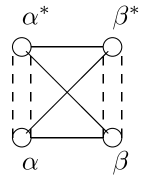
III. $[[E^\alpha, E^\beta], E^{\beta^*}] = 0$
$[[E^{-\alpha}, E^{-\beta}], E^{-\beta^*}] = 0$ for



IV. $[[[E^\alpha, E^\beta], E^\gamma], E^{\beta^*}] = 0$
$[[[E^{-\alpha}, E^{-\beta}], E^{-\gamma}], E^{-\beta^*}] = 0$ for



V. $[[E^{\alpha^*}, E^{-\alpha}], E^\beta] = E^{\beta^*}$
$[[E^{-\alpha^*}, E^\alpha], E^{-\beta}] = E^{-\beta^*}$ for



where h runs over $\tilde{\mathfrak{h}}$ in I, α, β run over $\pm\Gamma_{\text{ell}}$ in I, II, and α, β, γ run over $\pm\Gamma_{af}$ in III, IV and V.

We set $e_\alpha := E^\alpha$, $f_\alpha := -E^{-\alpha}$ for $\alpha \in \Gamma_{\text{ell}}$ (i.e. $e_\alpha^* := e_{\alpha^*} = E^{\alpha^*}$, $f_\alpha^* := f_{\alpha^*} = -E^{-\alpha^*}$), and set $a_{\alpha\beta} := I(\alpha^\vee, \beta)$, then the matrix $(a_{\alpha\beta})_{\alpha,\beta \in \Gamma_{\text{ell}}}$ is called the elliptic Cartan matrix. Now we normalize $I(\alpha, \alpha) = 2$ so that $\alpha^\vee = \alpha$, then using the above

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conventions, the defining relations are rewritten as follows:

Lemma 2.2. *The elliptic Lie algebra $\tilde{\mathfrak{g}}(R)$ is described by the following generators and relations.*

generators: $\tilde{\mathfrak{h}}$ *and* e_α, f_α *for* $\alpha \in \Gamma_{\mathrm{ell}}$

relations:

0. $\tilde{\mathfrak{h}}$ *is abelian*

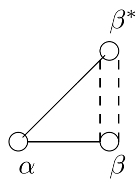
I. $[h, e_\alpha] = \langle h, \alpha \rangle e_\alpha$
$[h, f_\alpha] = -\langle h, \alpha \rangle f_\alpha$

II.1. $[e_\alpha, f_\alpha] = h_\alpha$
$[e_\alpha, f_\beta] = 0$ *if* $a_{\alpha\beta} \leq 0$

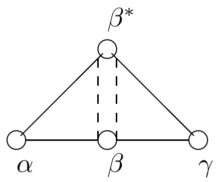
II.2. $[e_\alpha, e_\alpha^*] = 0$, $[f_\alpha, f_\alpha^*] = 0$

II.3. $(ade_\alpha)^{1-a_{\alpha\beta}} e_\beta = 0$ *if* $a_{\alpha\beta} \leq 0$
$(adf_\alpha)^{1-a_{\alpha\beta}} f_\beta = 0$ *if* $a_{\alpha\beta} \leq 0$

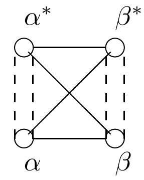
III. $ade_\beta^* ade_\beta e_\alpha = 0$
$adf_\beta^* adf_\beta f_\alpha = 0$ *for*



IV. $ade_\beta^* ade_\gamma ade_\beta e_\alpha = 0$
$adf_\beta^* adf_\gamma adf_\beta f_\alpha = 0$ *for*



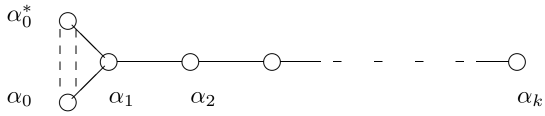
V. $ade_\beta ade_\alpha^* f_\alpha = e_\beta^*$
$adf_\beta adf_\alpha^* e_\alpha = f_\beta^*$ *for*



Remark 2.3. We have the relations$[h_\alpha, e_\beta] = a_{\alpha\beta} e_\beta$, $[h_\alpha, f_\beta] = -a_{\alpha\beta} f_\beta$.

3. The main theorem. We consider the extended elliptic diagram $\widetilde{\Gamma_{\mathrm{ell}}}$ consisting of all pairs of α_i, α_i^* $(0 \leq i \leq l)$, if necessary, by adding some vertices α_i^* to Γ_{ell}. In what follows, we consider $\widetilde{\Gamma_{\mathrm{ell}}}$ instead of Γ_{ell}. In the following diagram (3.1), we define $e_{\alpha_1}^* := ade_{\alpha_1} ade_{\alpha_0}^* f_{\alpha_0}$, $f_{\alpha_1}^* := adf_{\alpha_1} adf_{\alpha_0}^* e_{\alpha_0}$ and inductively $e_{\alpha_i}^*$, $f_{\alpha_i}^*$ for all added vertices α_i (see [5]),

(3.1)



Then from the results of [5] (Theorem 4.1 and its proof, i.e. from the realization of $\tilde{\mathfrak{g}}(R)$ by the vertex algebra and the relations of the corresponding elements in the vertex algebra), we can regard $\tilde{\mathfrak{g}}(R)$ as the Lie algebra generated by the elements e_α, e_α^*, f_α, f_α^* for $\alpha \in \{\alpha_0, \dots, \alpha_l\}$ with the relations in Lemma 2.2.

Lemma 3.1. *For* α, $\beta \in \{\alpha_0, \dots, \alpha_l\}$, *there hold the following relations.*

(i) $[e_\alpha, e_\beta^*] = [e_\alpha^*, e_\beta]$

(ii) $[f_\alpha, f_\beta^*] = [f_\alpha^*, f_\beta]$

Proof. (i) When $a_{\alpha\beta} \geq 0$, the two sides of the equation (i) vanish, and when $a_{\alpha\beta} = -1$, by using the relation V in Lemma 2.2,

$$\begin{aligned}
[e_\beta^*, e_\alpha] &= [ade_\beta ade_\alpha^* f_\alpha, e_\alpha] \\
&= [[e_\beta, [e_\alpha^*, f_\alpha]], e_\alpha] \\
&= -[[f_\alpha, [e_\beta, e_\alpha^*]], e_\alpha] \quad (\text{by } [f_\alpha, e_\beta] = 0) \\
&= [[e_\alpha, f_\alpha], [e_\beta, e_\alpha *]] \quad (\text{by } [[e_\beta, e_\alpha^*], e_\alpha] = 0) \\
&= [h_\alpha, [e_\beta, e_\alpha^*]] \\
&= -[e_\beta, [e_\alpha^*, h_\alpha]] - [e_\alpha^*, [h_\alpha, e_\beta]] \\
&= 2[e_\beta, e_\alpha^*] + [e_\alpha^*, e_\beta] \\
&= [e_\beta, e_\alpha^*]
\end{aligned}$$

so we get (i), and (ii) is similar. □

Theorem 3.2. *The elliptic Lie algebra $\tilde{\mathfrak{g}}(R)$ is described by the following generators and relations.*

generators: $\tilde{\mathfrak{h}}$ *and* e_α, f_α *for* $\alpha \in \widetilde{\Gamma_{\mathrm{ell}}}$

relations:

0. $\tilde{\mathfrak{h}}$ *is abelian*

I. $[h, e_\alpha] = \langle h, \alpha \rangle e_\alpha$
$[h, f_\alpha] = -\langle h, \alpha \rangle f_\alpha$

II.1. $[e_\alpha, f_\alpha] = h_\alpha$
$[e_\alpha, f_\beta] = 0$ *if* $a_{\alpha\beta} \leq 0$

II.2. $[e_\alpha, e_\alpha^*] = 0$, $[f_\alpha, f_\alpha^*] = 0$

II.3. $(ade_\alpha)^{1-a_{\alpha\beta}} e_\beta = 0$ *if* $a_{\alpha\beta} \leq 0$
$(adf_\alpha)^{1-a_{\alpha\beta}} f_\beta = 0$ *if* $a_{\alpha\beta} \leq 0$

III. $[e_\alpha^*, e_\beta] = [e_\alpha, e_\beta^*]$, $[f_\alpha^*, f_\beta] = [f_\alpha, f_\beta^*]$

where h runs over $\tilde{\mathfrak{h}}$ in I, *and α, β run over $\widetilde{\Gamma_{\mathrm{ell}}}$ in* I, II.1, II.3 *and run over Γ_{af} in* II.2, III.

Proof. It suffices to show that the relations III, IV, and V in Lemma 2.2 can be obtained from the relations in Theorem 3.2. We use the multi-bracket of length n ([5]),

$$[x_n, \ldots, x_3, x_2, x_1] := [x_n, [x_{n-1}, \cdots [x_3, [x_2, x_1]] \cdots]$$

and the following identity ([5]),
for $1 < s \leq n$,

$$\begin{aligned} & [y, x_n, \ldots, x_3, x_2, x_1] \\ = & [x_n, \ldots, x_{s+1}, x_s, y, x_{s-1}, \ldots, x_1] \\ & + [x_n, \ldots, x_{s+1}, [y, x_s], x_{s-1}, \ldots, x_1] \\ & + [x_n, \ldots, [y, x_{s+1}], x_s, x_{s-1}, \ldots, x_1] + \cdots \\ & \cdots + [[y, x_n], \ldots, x_{s+1}, x_s, x_{s-1}, \ldots, x_1]. \end{aligned}$$

III.
$$\begin{aligned} & ade^*_\beta ade_\beta e_\alpha \\ & = [e^*_\beta, e_\beta, e_\alpha] \\ & = [e_\beta, [e^*_\beta, e_\alpha]] + [[e_{\beta^*}, e_\beta], e_\alpha] \\ & = [e_\beta, [e_\beta, e^*_\alpha]] \quad \text{(by II.2, III)} \\ & = 0 \quad \text{(by II.3)} \end{aligned}$$

IV.
$$\begin{aligned} & ade^*_\beta ade_\gamma ade_\beta e_\alpha \\ & = [e^*_\beta, e_\gamma, e_\beta, e_\alpha] \\ & = [[e^*_\beta, e_\gamma], e_\beta, e_\alpha] + [e_\gamma, [e^*_\beta, e_\beta], e_\alpha] \\ & = [[e_\beta, e^*_\gamma], e_\beta, e_\alpha] \quad \text{(by II.2, III)} \\ & = [[e^*_\gamma, e_\beta], e_\alpha, e_\beta] \\ & = [e^*_\gamma, e_\beta, e_\alpha, e_\beta] - [e_\beta, [e^*_\gamma, e_\alpha], e_\beta] \\ & = 0 \end{aligned}$$

V.
$$\begin{aligned} & ade_\beta ade^*_\alpha f_\alpha \\ & = [e_\beta, e^*_\alpha], f_\alpha] + [e^*_\alpha, [e_\beta, f_\alpha]] \\ & = [[e^*_\beta, e_\alpha], f_\alpha] \\ & = [[f_\alpha, e_\alpha], e^*_\beta] \\ & = -[h_\alpha, e^*_\beta] \\ & = e^*_\beta \end{aligned}$$

so the proof is completed. □

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