A note on the Demjanenko matrices related to the cyclotomic $\mathbf{Z}_p$-extension

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Abstract: In this note, we define the Demjanenko matrices related to the cyclotomic $\mathbf{Z}_p$-extension, which can be regarded as generalization of the ordinary Demjanenko matrices. As a special case, we generalize the Maillet determinant defined by Girstmair.

Key words: Demjanenko matrices; Maillet determinants; $\mathbf{Z}_p$-extensions.

0. Introduction. Let $\mathbf{N}$ be the set of natural numbers, $\mathbf{Z}$ the ring of rational integers, $\mathbf{Q}$ the field of rational numbers, $\mathbf{R}$ the field of real numbers and $\mathbf{Z}_p$ the ring of p-adic integers for a prime p. For $n \in \mathbf{N}$, the Maillet determinant $D(n)$ can be defined by

$$D(n) = \det \left(R_n(ab') \right)_{a,b \in A_n},$$

where $A_n = \{a \in \mathbf{Z} \mid 1 \leq a < n/2,\ (a,n) = 1\}$, $R_n(x)$ is the residue of x modulo n with $0 \leq R_n(x) < n$, and x' is the integer with $xx' \equiv 1 \pmod{n}$ and $1 \leq x' < n$ for $x \in \mathbf{Z}$. $D(n)$ was conjectured not to be zero.

For an imaginary abelian field K, we define the relative class number $h^-(K)$ by $h^-(K) = h(K)/h(K \cap \mathbf{R})$, where $h(K)$ and $h(K \cap \mathbf{R})$ are the class numbers of K and $K \cap \mathbf{R}$, respectively.

Carlitz and Olson proved the following fascinating formula for any odd prime p:

$$D(p) = (-p)^{(p-3)/2}\ h^-(\mathbf{Q}(\zeta_p)), \tag{0.1}$$

where $h^-(\mathbf{Q}(\zeta_p))$ is the relative class number of the p-th cyclotomic field $\mathbf{Q}(\zeta_p)$. This fact showed that $D(p) \neq 0$ for any odd prime p. In the general case, Tateyama proved the generalized formula of (0.1), and gave the criterion whether $D(n) = 0$ or not (see [5]). The above formula (0.1) has been investigated by a lot of authors. Recently Girstmair defined a generalization of the Maillet determinant for imaginary abelian number fields and proved a generalized formula of (0.1) (see [3]). As an analogue of $D(p)$, the Demjanenko matrix $M(p)$ was defined by $M(p) = (c(ab))$ with $a, b \in A_p$, where

$$c(x) = \begin{cases} 1 & \text{if } R_p(x) \in A_p, \\ 0 & \text{otherwise,} \end{cases}$$

for $x \in \mathbf{Z}$. In [1], Hazama proved a relation between the Demjanenko matrix and the relative class number of $\mathbf{Q}(\zeta_p)$. This relation can be regarded as an analogue of (0.1). Hazama's result was generalized by Sands and Schwarz (see [4]). They defined the Demjanenko matrix for an imaginary abelian field of odd prime power conductor. They proved the relation formula between the determinant of their matrix and the relative class number of the field. Recently we succeeded in generalizing this result as follows (see [6]). Let K be an imaginary abelian field and n be its conductor. We can assume that $n \not\equiv 2 \pmod 4$. For $\ell \in \mathbf{Z}$ with $(\ell, n) = 1$ and $\ell > 1$, we defined the generalized Demjanenko matrix $\Delta(K, \ell)$ (see [6, Definition 2.5]). We proved the relation formula between $\det \Delta(K, \ell)$ and the relative class number $h^-(K)$, which could be regarded as a generalization of the one in [1] and [4]. In fact, we verified that $\Delta(K, 2)$ played the same role as the ordinary Demjanenko matrix. Moreover we verified that $\det \Delta(K, n+1)$ coincided with the Maillet determinant defined by Girstmair in [3]. Hence the result in [6] showed that the Maillet determinant and the Demjanenko matrix could be treated as an unity. In [2], Hirabayashi generalized our result completely. His result holds for any imaginary abelian field even if $\ell = 2$. Recently Kučera modified Hirabayashi's result. The above generalizations are natural, but the size of $\Delta(K, \ell)$ becomes larger as the degree $[K : \mathbf{Q}]$ becomes larger. It is not useful when the degree is large.

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In this note, we construct the Demjanenko matrices attached to the cyclotomic $\mathbf{Z}_p$-extension of an imaginary abelian field K:

$$K = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_m \subset \cdots,$$

for an odd prime p. We assume that the conductor of K is equal to dp such that $(d,p)=1$. Let $\ell \in \mathbf{Z}$ with $(\ell, dp) = 1$ and $\ell > 1$. Corresponding to $\{K_m \mid m \geq 0\}$, we define the Demjanenko matrices $\{\Delta(K,\ell,m) \mid m \geq 0\}$ (see Definition 2.5). Then the following theorem holds.

Theorem. *For $m \geq 0$,*

$$\det \Delta(K,\ell,m) = \frac{(-2)^{[K_m:Q]/2}}{Q(K_m)w(K_m)} h^-(K_m) \times \prod_{\chi \in X_m^-} \left((\ell\chi(\ell) - 1) \prod_{\substack{q:\text{prime} \\ q \mid dp}} (1 - \chi(q)) \right),$$

where $Q(K_m)$ is what is called the unit index of K_m, $w(K_m)$ is the number of roots of unity in K_m, and X_m^- is the set of odd primitive characters of $\mathrm{Gal}(K_m/\mathbf{Q})$.

On the above assumption, we can see that $\Delta(K,\ell,0) = \Delta(K,\ell)$ and

$$\Delta(K,\ell,m) \in \mathrm{M}\left(\frac{[K:\mathbf{Q}]}{2}, \mathbf{Q}\right),$$

for any $m \in \mathbf{N}$ (see Lemma 2.6).

1. Preliminaries. We make use of the same notations as in Chap. 7 of [7]. In this section, we fix $m \in \mathbf{Z}$ with $m \geq 0$. Let

$$\begin{aligned}&\mathrm{Gal}(\mathbf{Q}(\zeta_{dp^{m+1}})/\mathbf{Q}) \\ &\quad = \{\sigma_a \mid \sigma_a : \zeta_{dp^{m+1}} \to \zeta_{dp^{m+1}}^a,\ (a,dp)=1\},\end{aligned}$$

where $\zeta_n = \exp(2\pi i/n)$. Since $K_m \subset \mathbf{Q}(\zeta_{dp^{m+1}})$, we let σ_a denote both the element of $\mathrm{Gal}(\mathbf{Q}(\zeta_{dp^{m+1}})/\mathbf{Q})$ and its restriction to K_m. Let $G_K = \mathrm{Gal}(K/\mathbf{Q})$, $\Gamma_m = \mathrm{Gal}(K_m/K)$ and $G_m = \mathrm{Gal}(K_m/\mathbf{Q})$. Then we have $G_m \simeq G_K \times \Gamma_m$. Corresponding to this decomposition, we can write $\sigma_a = \delta(a)\gamma_m(a)$, where $\delta(a) \in G_K$ and $\gamma_m(a) \in \Gamma_m$. Let $J = \sigma_{-1}$ be complex conjugation. We consider $\overline{\mathbf{Q}}$ which is an algebraic closure of $\mathbf{Q}$, and consider the group ring $V = \overline{\mathbf{Q}}[G_K]$. Let $V^- = \{x \in V \mid Jx = -x\}$. We can see that $V^- = (1-J)V$.

Since $K \subset \mathbf{Q}(\zeta_{dp})$, we can take

$$T_K \subset \{a \in \mathbf{Z} \mid 1 \leq a < dp,\ (a,dp)=1\},$$

such that $G(\mathbf{Q}(\zeta_{dp})/K) = \{\sigma_a \mid a \in T_K\}$. Since $K/\mathbf{Q}$ is an imaginary abelian extension, we can uniquely take a set

$$S_K \subset \{c \in \mathbf{Z} \mid 1 \leq c < dp/2,\ (c,dp)=1\}$$

such that

$$G_K = \{\sigma_c \mid c \in S_K\} \cup \{\sigma_{-c} \mid c \in S_K\}.$$

Let Θ, Y_m and X_m be the character groups of G_K, Γ_m and G_m respectively. And let Θ^- and X_m^- be sets of odd characters in Θ and X_m respectively. For $\chi \in X_m^-$, we may uniquely write $\chi = \theta\psi$, where $\theta \in \Theta^-$ and $\psi \in Y_m$. Then θ is a character of conductor dividing d or dp, while ψ has p-power order and is either trivial or has conductor of the form p^j. θ (resp. ψ) is called a character of the first (resp. second) kind. Note that the characters of the first kind are associated with $\mathbf{Q}(\zeta_{dp})$, while those of the second kind are associated with the subfield of $\mathbf{Q}(\zeta_{dp^{m+1}})$ of degree p^m over $\mathbf{Q}$. Hence the characters of the first kind correspond to tame ramification at p, while those of the second kind correspond to wild ramification. We see that ψ is an even character since it corresponds to a real field. So we can see that if χ is even then θ is even.

Let

$$A_n(b,\ell) = \sum_{\substack{\zeta^\ell = 1 \\ \zeta \neq 1}} \frac{\zeta^{n-b}}{1-\zeta^n} \in \mathbf{Q}, \tag{1.1}$$

for $b \in \mathbf{Z}$. For simplicity, we denote $A(b,\ell)$ instead of $A_{dp^{m+1}}(b,\ell)$. Let $\overline{\chi} = \chi^{-1}$. For $\psi \in Y_m$, we define

$$\begin{aligned}\rho_\psi &= \rho_\psi(K_m,\ell) \\ &= \sum_{\substack{a=1 \\ (a,dp)=1}}^{dp^{m+1}} A(a,\ell)\overline{\psi}(a)\delta(a)^{-1} \in V.\end{aligned} \tag{1.2}$$

Lemma 1.1. *$\rho_\psi \in V^-$ for any $\psi \in Y_m$.*

For $\theta \in \Theta^-$, we consider the orthogonal idempotent of V^-

$$\varepsilon_\theta = \frac{1}{[K:\mathbf{Q}]} \sum_{a \in S_K} \theta(a)\ (\delta(a)^{-1} - \delta(-a)^{-1}).$$

Note that $\varepsilon_\theta \delta(a)^{-1} = \overline{\theta}(a)\varepsilon_\theta$. We can easily verify that $\{\varepsilon_\theta \mid \theta \in \Theta^-\}$ forms a $\overline{\mathbf{Q}}$-basis for V^-. For $r \in V^-$, let L_r be the endomorphism of V^- defined by $L_r(v) = rv$. By [6, Lemma 1.2], we get the following.

Lemma 1.2. *For $\theta \in \Theta^-$ and $\psi \in Y_m$,*

$$L_{\rho_\psi}(\varepsilon_\theta) = \varepsilon_\theta \Big[(\ell\overline{\theta\psi}(\ell) - 1) \times B_{1,\overline{\theta\psi}} \prod_{q|dp} (1 - \overline{\theta\psi}(q)) \Big],$$

where $B_{1,\chi}$ is the generalized Bernoulli number (see e.g. [7] Chap. 4).

It seems that the values $\{A(b,\ell)\}$ are unclear. So we need to make these values clear. In fact we can prove the following lemma.

Lemma 1.3. *Let $\widetilde{R}(\cdot) = R_{dp^{m+1}}(\cdot)$. With the above notations,*

$$A(\widetilde{R}(a), \ell) = \frac{\ell\widetilde{R}(a\ell') - \widetilde{R}(a)}{dp^{m+1}} - \frac{\ell - 1}{2}.$$

Proof. Let $\eta = \sum_a A(\widetilde{R}(a), \ell)\sigma_a^{-1} \in V$, then we can see that $\eta \in V^-$ by the proof of Lemma 1.1. By [6, Lemma 1.2], we have

$$\eta\varepsilon_\chi = (\ell\overline{\chi}(\ell) - 1)B_{1,\overline{\chi}}\varepsilon_\chi \prod_{q|dp}(1 - \overline{\chi}(q)), \tag{1.3}$$

for any $\chi \in X_m^-$. On the other hand, for the Bernoulli polynomial $B_1(x) = x - 1/2$, let

$$\tau = (\ell\sigma_\ell^{-1} - 1) \sum_{\substack{a=1 \\ (a,dp)=1}}^{dp^{m+1}} B_1\left(\frac{\widetilde{R}(a)}{dp^{m+1}}\right)\sigma_a^{-1},$$
$$= \sum_{\substack{a=1 \\ (a,dp)=1}}^{dp^{m+1}} \left\{ \frac{\ell\widetilde{R}(a\ell') - \widetilde{R}(a)}{dp^{m+1}} - \frac{\ell-1}{2} \right\} \sigma_a^{-1},$$

where l' is the integer with $ll' \equiv 1 \pmod{dp^{m+1}}$ and $1 \le l' < dp^{m+1}$. Then it follows from (1.3) that $\tau\varepsilon_\chi = \eta\varepsilon_\chi$ for any $\chi \in X_m^-$. Since τ, $\eta \in V^-$, we have $\tau = \eta$ in V. Thus we get the proof. □

2. Definition of $\boldsymbol{\Delta(K, \ell, m)}$. For $a \in \mathbf{Z}$ with $(a, dp) = 1$, let

$$\xi(a) = \frac{\delta(a)^{-1} - \delta(-a)^{-1}}{2}.$$

A short calculation shows that $\xi(a)\xi(b) = \xi(ab)$ and $\xi(-a) = -\xi(a)$. We can also verify that $\{\xi(s) \mid s \in S_K\}$ forms a $\overline{\mathbf{Q}}$-basis for V^-. In this section, we shall determine the matrix of L_{ρ_ψ} with respect to $\{\xi(s) \mid s \in S_K\}$.

Note that $\delta(b + dpk) = \delta(b)$. By Lemma 1.1, we get

$$\rho_\psi = \sum_{\substack{a=1 \\ (a,dp)=1}}^{dp} \sum_{j=0}^{p^m - 1} A(a + dpj, \ell)\overline{\psi}(a + dpj)\delta(a)^{-1}. \tag{2.1}$$

For $a \in \mathbf{Z}$, let $R(a)$ be the residue of a modulo dp with $0 \le R(a) < dp$, and let a' be the integer with $aa' \equiv 1 \pmod{dp}$ and $1 \le a' < dp$. By the definition of S_K and T_K, we see that

$$\{R(ts) \mid s \in S_K, t \in T_K\} \cup \{R(-ts) \mid s \in S_K, t \in T_K\}$$

forms a set of representatives for $(\mathbf{Z}/dp\mathbf{Z})^\times$. For simplicity, we let

$$\beta(c, dp^{m+1}, \ell, \psi) = \sum_{j=0}^{p^m - 1} A(R(c) + dpj, \ell)\overline{\psi}(R(c) + dpj),$$

for $c \in \mathbf{Z}$. Since $\delta(R(ts)) = \delta(s)$ and $\delta(R(-ts)) = \delta(-s)$ for $s \in S_K$ and $t \in T_K$, we have

$$\rho_\psi = \sum_{s \in S_K} \sum_{t \in T_K} \Big\{ \beta(ts, dp^{m+1}, \ell, \psi)\delta(s)^{-1} + \beta(-ts, dp^{m+1}, \ell, \psi)\delta(-s)^{-1} \Big\}. \tag{2.2}$$

Lemma 2.1. *With the above notations,*

$$\beta(-ts, dp^{m+1}, \ell, \psi) = -\beta(ts, dp^{m+1}, \ell, \psi).$$

Proof. The left-hand side of above equation is equal to

$$\sum_{j=0}^{p^m - 1} A(dp - R(ts) + dpj, \ell) \times \overline{\psi}(dp - R(ts) + dpj). \tag{2.3}$$

By the facts that $A(dp^{m+1} - a, \ell) = -A(a, \ell)$ and $\psi(dp^{m+1} - a) = \psi(a)$, we can see that (2.3) is equal to the right-hand side of above equation by letting $k = p^m - 1 - j$. Thus we have the assertion. □

By (2.2) and Lemma 2.1, we have

$$\rho_\psi = \sum_{s \in S_K} \left(2 \sum_{t \in T_K} \beta(ts, dp^{m+1}, \ell, \psi) \right) \xi(s). \tag{2.4}$$

In order to determine the matrix of L_{ρ_ψ} with respect to the basis $\{\xi(s)\}$, we recall the following two functions $f(x)$ and $g(x)$ (see [6] §2). For $a \in \mathbf{Z}$ with $(a, dp) = 1$, let $g(a) = R(a)$ and $f(a) = 1$ if $1 \le R(a) < dp/2$, and $g(a) = dp - R(a)$ and $f(a) = -1$ if $dp/2 < R(a) < dp$. We can immediately see that

$1 \leq g(a) < dp/2$ and $g(a)f(a) \equiv a \pmod{dp}$. The following two lemmas were proved (see [6, Lemma 2.2, Lemma 2.3]).

Lemma 2.2. *If $r \in S_K$, then $\{g(sr) \mid s \in S_K\} = S_K$.*

Lemma 2.3. *Let $s, r, u \in S_K$ with $g(sr) = u$. Then $s = g(ur')$ and $\xi(sr) = f(ur')\,\xi(u)$.*

Proposition 2.4. *For $r \in S_K$,*

$$L_{\rho_\psi}(\xi(r)) = \sum_{s \in S_K} \left(2 \sum_{t \in T_K} \beta(tsr', dp^{m+1}, \ell, \psi) \right) \xi(s).$$

Proof. It follows from (2.4) that

$$L_{\rho_\psi}(\xi(r)) = \sum_{s \in S_K} \left(2 \sum_{t \in T_K} \beta(ts, dp^{m+1}, \ell, \psi) \right) \xi(sr). \tag{2.5}$$

Let $g(sr) = u$. It follows from Lemma 2.3 that $s = g(ur')$ and $\xi(sr) = f(ur')\xi(u)$. By Lemma 2.2, we see that the right-hand side of (2.5) is equal to

$$\sum_{u \in S_K} \left(2 \sum_{t \in T_K} \beta(tg(ur'), dp^{m+1}, \ell, \psi) \right) \times f(ur')\xi(u). \tag{2.6}$$

If $f(ur') = 1$ then $g(ur') = R(ur')$, so $R(tg(ur')) = R(tur')$. If $f(ur') = -1$ then $g(ur') = dp - R(ur') = R(-ur')$, so $R(tg(ur')) = R(-tur')$. In the both cases, it follows from Lemma 2.1 that (2.6) is equal to

$$\sum_{u \in S_K} \left(2 \sum_{t \in T_K} \beta(tur', dp^{m+1}, \ell, \psi) \right) \xi(u).$$

Thus we have the assertion. □

Definition 2.5.

$$\Delta_\psi(K, \ell, m) = \left(2 \sum_{t \in T_K} \sum_{j=0}^{p^m - 1} A(R(tsr') + dpj, \ell) \times \overline{\psi}(R(tsr') + dpj) \right)_{s, r \in S_K},$$

for $\psi \in Y_m$, and

$$\Delta(K, \ell, m) = \prod_{\psi \in Y_m} \Delta_\psi(K, \ell, m),$$

for $m \geq 0$, where $\prod$ means the matrix product.

Remark. $\Delta(K, \ell, m)$ can be regarded as a generalization of the ordinary Demjanenko matrix $\Delta(K, \ell)$. In fact, we can easily verify that $\Delta(K, \ell, 0)$ coincides with $\Delta(K, \ell)$, if the conductor of K is equal to dp with $(d, p) = 1$ (see [6, Definition 2.5]). This assumption is important. Hirabayashi and the referee pointed out the fact that $\Delta(K, \ell, 0)$ did not necessarily coincide with $\Delta(K, \ell)$, if the conductor of K was equal to d with $(d, p) = 1$.

Lemma 2.6. *For any $m \geq 0$,*

$$\Delta(K, \ell, m) \in \mathrm{M}\left(\frac{[K : \mathbf{Q}]}{2}, \mathbf{Q} \right).$$

Proof. For any $\sigma \in \mathrm{Gal}(\mathbf{Q}(\zeta_{p^m})/\mathbf{Q})$, we have $\{\psi^\sigma \mid \psi \in Y_m\} = Y_m$. For a matrix $C = (c_{ij})$ with $c_{ij} \in \mathbf{Q}(\zeta_{p^m})$, let $C^\sigma = (c_{ij}^\sigma)$. Then $\Delta(K, \ell, m)^\sigma = \prod_\psi \Delta_{\psi^\sigma}(K, \ell, m) = \Delta(K, \ell, m)$ for any σ. Thus we have the assertion. □

3. Proof of Theorem and some examples.

By Proposition 2.4, we see that $\Delta_\psi(K, \ell, m)$ is the matrix of L_{ρ_ψ} with respect to $\{\xi(s) \mid s \in S_K\}$, for any $\psi \in Y_m$. By combining Lemma 1.2, Proposition 2.4 and Definition 2.5, we have

$$\det \Delta(K, \ell, m) = \prod_{\chi \in X_m^-} (\ell\chi(\ell) - 1) \times B_{1,\chi} \prod_{q \mid dp} (1 - \chi(q)).$$

By using the analytic class number formula, we get the proof of Theorem.

Example. Let $p = 5$, $d = 1$ and $K = K_0 = \mathbf{Q}(\zeta_5)$. So $K_1 = \mathbf{Q}(\zeta_{25})$. We can take $\ell = 2$. Since $A(b, 2) = (-1)^{b+1}/2$ for $b \in \mathbf{Z}$, we can calculate that

$$\Delta(\mathbf{Q}(\zeta_5), 2, 1) = \begin{pmatrix} 16 & -144 \\ 144 & 16 \end{pmatrix}$$

We can verify that $\det \Delta(\mathbf{Q}(\zeta_5), 2, 1) = 20992$, which is equal to

$$\frac{1}{w(\mathbf{Q}(\zeta_{25}))} (-2)^{[Q(\zeta_{25}):Q]/2} \times h^-(\mathbf{Q}(\zeta_{25})) \prod_{\chi \in X_1^-} (2\chi(2) - 1).$$

We consider the case $\ell = dp^{m+1} + 1$. By [6, Eq. (3.3)], we have

$$\sum_{a \in T_K} A\left(\widetilde{R}(ac), dp^{m+1} + 1\right) = \sum_{a \in T_K} dp^{m+1}\, B_1\left(\frac{\widetilde{R}(ac)}{dp^{m+1}} \right),$$

for $c \in S_K$. Let $D_\psi(K, m) = \det \Delta_\psi(K, dp^{m+1} +$

$1, m)$. By Definition 2.5, we have the following.

Lemma 3.1. *For $\psi \in Y_m$,*

$$D_\psi(K,m) = \text{determinant of} \left(2dp^{m+1} \sum_{t \in T_K} \sum_{j=0}^{p^m-1} B_1\left(\frac{R(tsr')+dpj}{dp^{m+1}}\right) \times \overline{\psi}(R(tsr')+dpj)\right)_{s,r \in S_K}.$$

Hence we define

$$D(K,m) = \prod_{\psi \in Y_m} D_\psi(K,m),$$

for $m \geq 0$. We can regard those as the Maillet determinants attached to the cyclotomic $\mathbf{Z}_p$-extension of K. Note that $D(K,0)$ coincides with the Maillet determinant $D^*(K)$ attached to K defined in [3], if the conductor of K is equal to dp with $(d,p)=1$. By applying Theorem, we get the following.

Proposition 3.2. *For $m \geq 0$,*

$$D(K,m) = \frac{(-2dp^{m+1})^{[K_m:Q]/2}}{Q(K_m)\omega(K_m)} h^-(K_m) \times \prod_{\chi \in X_m^-} \prod_{q \mid dp} (1-\chi(q)).$$

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