Global existence of solutions to the Proudman–Johnson equation

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(Communicated by Heisuke HIRONAKA, M. J. A., Nov. 13, 2000)

Abstract: We show that there is no blow-up solutions, for positive viscosity constant ν, to the equation $f_{xxt} - \nu f_{xxxx} + f f_{xxx} - f_x f_{xx} = 0$, $x \in (0,1)$, $t > 0$ with (i) periodic boundary condition, or (ii) Dirichlet boundary condition $f = f_x = 0$ or (iii) Neumann boundary condition $f = f_{xx} = 0$ on the boundary $x = 0, 1$. Furthermore we show that every solution decays to the trivial steady state as t goes to infinity.

Key words: Proudman–Johnson equation; global existence.

1. Introduction. We consider the Proudman–Johnson equation, for $f = f(x,t)$,

$$(1.1)\quad f_{xxt} - \nu f_{xxxx} + f\, f_{xxx} - f_x\, f_{xx} = 0, \quad x \in (0,1), \quad t > 0$$

with one of the following homogeneous boundary conditions:

(PBC) f is periodic in x and $\int_0^1 f(x,t)dx = 0$;

(DBC) $f(0,t) = f(1,t) = f_x(0,t) = f_x(1,t) = 0$;

(NBC) $f(0,t) = f(1,t) = f_{xx}(0,t) = f_{xx}(1,t) = 0$.

Here $\nu > 0$ is a constant called viscosity, t the time variable, x the space, and subscripts stand for differentiation.

Equation (1.1) is derived from the Navier–Stokes equations by assuming a special form of the velocity field $\mathbf{u}(x,y,t) = (f, -yf_x)$ and considering the flow in a two-dimensional channel; see [5, 9, 13, 8, 1, 3, 4, 7, 12] and the references therein. The boundary condition (DBC) corresponds to the Dirichlet boundary condition $\mathbf{u} = 0$. The boundary condition (NBC), having its own meaning in the Navier–Stokes flow, can also be derived from a magneto-hydrodynamic flow (see, for example, [6]). The periodic boundary condition (PBC) is often used in numerical simulations.

Childress *et al.* [1] in 1989 first reported that numerical solutions to (1.1) and (DBC) with large initial data blow-up in finite time. Later Cox [3] clarified that grid refinement could remove singularities. Grundy and McLaughlin [4], on the other hand, considered a non–homogeneous version of (NBC), demonstrated numerically the existence of blow-up solutions, and also derived asymptotically the singular behavior of the solutions near blow-up time. Recently, Okamoto and Shōji [10, 11] and Zhu [14] claimed that they cannot find any numerical blow–up solutions of (1.1) and (DBC). Very recently, Okamoto and Zhu [12] provided strong evidence showing that there is no blow-up for (1.1) with the homogeneous boundary conditions. Since the work of Childress et al. [1], it has been asked repeatedly whether solutions to (1.1) and (DBC) blow-up in finite time or not (see, for example, Constantin [2]). The purpose of this paper is to answer such a question.

Theorem. *For any* $f_0 \in C^4([0,1])$ ($\int_0^1 f_0(x)dx = 0$ *in case of* (PBC)), *there exists a unique solution* $f(x,t) \in C^\infty([0,1]\times(0,\infty))$ *to* (1.1) *with initial data* $f(\cdot,0) = f_0(\cdot)$ *and one of the homogeneous boundary conditions* (PBC), (DBC) *or* (NBC)*. In addition,*

$$\lim_{t\to\infty} f(\cdot,t) = 0.$$

Remark. In a forthcoming paper, we shall prove the global (in time) existence of solutions for the generalized Proudman–Johnson equation (see [12] for the derivation)

$$f_{xxt} - f_{xxxx} + f\, f_{xxx} - af_x f_{xx} = 0$$

1991 Mathematics Subject Classification. 35K55, 35Q30, 76D03.

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Chen was partially supported by the National Science Foundation grants DMS–9971043 and would like to acknowledge the support and hospitality provided by the Research Institute for Mathematical Science, Kyoto University. Okamoto was partially supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports and Culture of Japan, No. 11304005.

for a real parameter $a \in [-3,1]$ with (PBC) or (NBC). When $a \notin [-3,1]$, it is expected that solutions can blow up.

2. Proof of the Theorem.

Step 1. For smooth initial data, say in $C^2([0,1])$, local (in time) existence of unique solution to (1.1) with one of the boundary conditions (PBC), (DBC) or (NBC) follows from a standard parabolic theory. Also, via energy estimates (cf. examples in Step 5), one can show that all the higher order derivatives are bounded by the initial data and $\max|f|$. Hence, to prove the global existence, we need only estimate, a priori, $\max|f|$.

We assume that $f_0 \not\equiv 0$, so that $f(\cdot,t) \not\equiv 0$ for all t in its existence interval. We use $\|\cdot\|_p$ to denote the $L^p((0,1))$ norm, for $1 \le p \le \infty$.

Step 2. First we show that $\max|f|$ can be estimated by the minimum of f_{xxx}. For this, we introduce

$$m(t) = \min_{x\in[0,1]}\{f_{xxx}(x,t)\}.$$

(i) If we have (PBC) or (NBC) boundary condition, then $\int_0^1 f_{xxx}dx = 0$ and

$$\begin{aligned}\|f_{xx}(\cdot,t)\|_\infty &\le \int_0^1 |f_{xxx}(x,t)|\,dx\\ &= \int_0^1 \Big(|f_{xxx}| - f_{xxx}\Big)\,dx \le -2m(t).\end{aligned}$$

As a function of x, $\pm f - (1/2)\|f_{xx}\|_\infty x(1-x)$ is convex (second order derivative non-negative) and has boundary value zero, so it must be non–positive. Hence,

$$\|f(\cdot,t)\|_\infty \le \frac{1}{8}\|f_{xx}(\cdot,t)\|_\infty \le -\frac{1}{4}m(t).$$

(ii) Next we consider the (DBC) boundary condition. In this case, as a function of x, $f_x + (1/2)m(t)x(1-x)$ is convex and has boundary value zero, so it is non–positive. Thus, $f_x \le -(1/2)m(t)x(1-x)$. Consequently,

$$\begin{aligned}\|f(\cdot,t)\|_\infty &\le \int_0^1 |f_x|\,dx = \int_0^1 \Big(|f_x| + f_x\Big)dx\\ &\le -\frac{1}{6}m(t).\end{aligned}$$

Step 3. Now we show that $m(t)$ is bounded from below, uniformly in t.

Differentiating (1.1) with respect to x and setting $u = f_{xxx}$ we obtain

$$u_t - \nu u_{xx} + f\,u_x = f_{xx}^2 \ge 0. \tag{2.1}$$

Thus, a maximum principle implies that the minimum of u can only be obtained at the parabolic boundary; namely, for any $0 \le \tau < t$,

$$\begin{aligned}&\min_{[0,1]\times[\tau,t]}\{u\}\\ &= \min\Big\{\min_{[0,1]}\{u(\cdot,\tau)\},\ \min_{s\in[\tau,t]}\{u(0,s),u(1,s)\}\Big\}.\end{aligned}$$

In order to estimate $m(t)$, it remains to show that u cannot obtain its minimum on the lateral parabolic boundary.

(i) Suppose we have (PBC), the periodic boundary condition. Then by shifting the spatial region $x \in (0,1)$ if necessary, we conclude that u cannot obtain its minimum at the lateral parabolic boundary so that

$$m(t) > m(\tau) \ge m(0) = \min_{[0,1]}\{f_{xxx}(\cdot,0)\} \quad \forall\, t > \tau \ge 0. \tag{2.2}$$

(ii) Next, suppose we have (NBC), the Neumann boundary condition. From (1.1), we see that $u_x = f_{xxxx} = 0$ at the boundary $x = 0$, 1, so that the minimum of u cannot be obtained at the lateral parabolic boundary. Hence, (2.2) still holds.

(iii) Finally we consider (DBC), the Dirichlet boundary condition. For this, we need more work than that in the early cases.

Integrating (1.1) over x gives

$$f_{xt} - \nu f_{xxx} + f\,f_{xx} - f_x^2 + \gamma(t)\nu = 0 \tag{2.3}$$

where the combined $\gamma\nu$ is an integration constant. Upon evaluating the equation at the boundary $x = 0$ and $x = 1$, we see that

$$\gamma(t) = f_{xxx}(0,t) = f_{xxx}(1,t). \tag{2.4}$$

(Childress *et al.* [1] already observed this relation.) Now integrating (2.3) over $(0,1)$ and using $\int_0^1 f_x(x,t)dx = 0$ we obtain

$$\int_0^1 \{\gamma - f_{xxx}(x,t)\}\,dx = 2\nu^{-1}\int_0^1 f_x^2 > 0. \tag{2.5}$$

This equation, together with (2.4), shows that the minimum of f_{xxx} is strictly smaller than γ, the boundary value of f_{xxx}. Therefore, (2.2) still holds.

Combining estimates in Steps 2 and 3, we obtain an a priori estimate

$$\|f(\cdot,t)\|_\infty \le -\frac{1}{4}m(t) \le -\frac{1}{4}\min_{[0,1]}\{f_{xxx}(x,0)\} \tag{2.6}$$

for all t in its existence interval. From this, we see that any solution will not blow up, and exists globally in t.

Step 4. Now we show that any solution decays to zero as $t \to \infty$. Step 3 shows that $m(t)$ is an increasing and negative function. Hence $m(\infty) := \lim_{t\to\infty} m(t)$ exists. In view of (2.6), we need only show that $m(\infty) = 0$.

Suppose, to the contrary, that $m(\infty) < 0$. We shall show in Step 5 that $\{f(\cdot,t)\}_{t>1}$ is a bounded family in $C^4([0,1])$. Hence, we can find a sequence $\{t_j\}_{j=1}^\infty$ such that as $j \to \infty$, $t_j \to \infty$ and $f(\cdot,t_j) \to f_0^*$ in $C^3([0,1])$ for some smooth $f_0^* \not\equiv 0$. Then as $j \to \infty$, $m(t_j + t) \to m^*(t)$ for all finite $t \ge 0$ where $m^*(t) = \min_{[0,1]} f^*_{xxx}(\cdot,t)$ and f^* is the solution to (1.1) with initial data f_0^*. As f_0^* is non–trivial, $m^*(1) - m^*(0) > 0$. This contradicts the fact that $m^*(1) - m^*(0) = \lim_{j\to\infty}(m(t_j+1) - m(t_j)) = 0$. Thus, $f \to 0$ as $t \to \infty$.

Step 5. It remains to show that $\{f(\cdot,t)\}_{t>1}$ is a bounded family in $C^4([0,1])$, which follows from energy estimates. Here we provide, as an example, the estimate from $\|f\|_{L^\infty(0,1)\times(0,\infty)}$ to $\sup_{t>0}\|f_x(\cdot,t)\|_2$ and $\sup_{t>0}\|f_{xx}(\cdot,t)\|_2$.

First multiplying (1.1) by $2f$ and integrating the resulting equation over $x \in (0,1)$, we obtain, after integration by parts and using the boundary conditions,

$$\frac{d}{dt}\|f_x\|_2^2 + 2\nu\|f_{xx}\|_2^2 = -3\int_0^1 f\, f_x f_{xx}. \tag{2.7}$$

Since $\|f f_x\|_2^2 = (1/3)\|f^3 f_{xx}\|_1 \le (1/3)\|f\|_6^3\,\|f_{xx}\|_2$,

$$\begin{aligned}3\|f\, f_x f_{xx}\|_1 &\le 3\|f f_x\|_2\|f_{xx}\|_2 \le \sqrt{3}\|f\|_6^{3/2}\|f_{xx}\|_2^{3/2}\\ &\le \nu\|f_{xx}\|^2 + \nu^{-3}\|f\|_6^6.\end{aligned}$$

Therefore,

$$\frac{d}{dt}\|f_x\|_2^2 + \nu\|f_{xx}\|_2^2 \le \nu^{-3}\|f\|_6^6.$$

Multiplying both sides by $e^{\nu t}$ and using $\|f_x\|_2 \le \|f_{xx}\|_2$ yields $d/dt(e^{\nu t}\|f_x\|_2^2) \le \nu^{-3}e^{\nu t}\|f\|_6^6$, which gives, after integration,

$$\|f_x(\cdot,t)\|_2^2 \le \nu^{-4}\|f\|^6_{L^\infty([0,1]\times[0,\infty))} + e^{-\nu t}\|f_x(\cdot,0)\|_2^2.$$

This gives a bound on $\sup_{t\ge0}\|f_x(\cdot,t)\|_2$.

In a similar manner, e.g., multiplying the differential equation for $\partial_x^k f$ by $\partial_x^k f$, $k = 2,3,\ldots$, one can establish estimates, uniform in t, for high order derivatives. When the boundary conditions are either (PBC) or (NBC), the proof is standard and is omitted. Here we provide only an estimate for f_{xx}, when the boundary conditions are (DBC).

Denote $\overline{f_{xxx}} = \int_0^1 f_{xxx}dx$ and $\overline{f_x^2} = \int_0^1 f_x^2 dx$, the average of f_{xxx} and f_x^2 respectively. Then the equation in (2.5) reads $\gamma = \overline{f_{xxx}} + 2\nu^{-1}\overline{f_x^2}$. Multiplying both sides of (2.3) by $2(\overline{f_{xxx}} - f_{xxx})$ and integrating over $x \in (0,1)$, we obtain, after integrating by parts and using $f_x = 0$ on the boundary,

$$\begin{aligned}\frac{d}{dt}\|f_{xx}\|_2^2 &+ 2\nu\|f_{xxx} - \overline{f_{xxx}}\|_2^2\\ &= \int_0^1 \Big(f_{xxx} - \overline{f_{xxx}}\Big)\Big(f f_{xx} - f_x^2 + 2\overline{f_x^2}\Big).\end{aligned} \tag{2.8}$$

Note that $\|f_{xx}\|_2^2 = -\int_0^1 f_x(f_{xxx} - \overline{f_{xxx}}) \le \|f_x\|_2\|f_{xxx} - \overline{f_{xxx}}\|_2$ and $\|f_x\|_2^2 \le \|f\|_2\|f_{xx}\|_2$, so that

$$\|f_{xx}\|_2 \le \|f\|_2^{1/3}\|f_{xxx} - \overline{f_{xxx}}\|_2^{2/3}.$$

In a similar manner, the right-hand side of (2.8) is bounded by $4\|f\|_\infty^{5/3}\|f_{xxx} - \overline{f_{xxx}}\|_2^{4/3}$. Hence, after using Young's inequality, we obtain

$$\frac{d}{dt}\|f_{xx}\|^2 + \nu\|f_{xx}\|_2^2 \le C(\nu, \|f\|_\infty).$$

Same as before, multiplying both sides by $e^{\nu t}$ and integrating over $(0,t)$ gives

$$\|f_{xx}\|_2^2 \le e^{-\nu t}\|f_{xx}(\cdot,0)\|_2^2 + C(\|f\|_{L^\infty((0,1)\times(0,t))}).$$

The higher order derivatives can be similarly estimated. We omit the details. This concludes the proof.

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