A twisted invariant for finitely presentable groups

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Abstract: Following Wada's construction [4] on the twisted Alexander polynomial, we introduce a new twisted invariant for finitely presentable groups.

Key words: Twisted invariant; finitely presentable group; Alexander polynomial.

1. Introduction. Let Γ be a finitely presentable group with a homomorphism α to a finite cyclic group $\mathbf{Z}/m$. We assume that the class number h_m of the m-th cyclotomic field $K = \mathbf{Q}(\zeta_m)$ is equal to one, where ζ_m is a primitive m-th root of unity. To each linear representation

$$\rho : \Gamma \to GL(n, \mathbf{Z})$$

of the group Γ we will assign an algebraic number $\Theta_{\Gamma,\rho} \in K$ called a twisted invariant of Γ associated to ρ. This is well-defined up to a factor of a unit of the ring $O_K = \mathbf{Z}[\zeta_m]$ of algebraic integers and is in fact an invariant of the group Γ, the associated homomorphism α and the representation ρ. Namely, although we need a presentation of Γ to define an algebraic number $\Theta_{\Gamma,\rho}$, yet it can be shown that

Theorem 1. *The twisted invariant $\Theta_{\Gamma,\rho} \in \mathbf{Q}(\zeta_m)$ is independent of the choice of the presentation.*

The construction of $\Theta_{\Gamma,\rho}$ and the proof of its invariance are based on the idea of Wada's paper [4], which introduces the twisted Alexander polynomial for finitely presentable groups with a homomorphism to a free abelian group. A merit of our framework here is that we can deal with finitely presentable groups such as the modular group and the orbifold fundamental group.

2. Construction. For a given homomorphism $\alpha : \Gamma \to \mathbf{Z}/m = \langle q \mid q^m \rangle$, we denote the induced homomorphism of the integral group ring by

$$\tilde{\alpha} : \mathbf{Z}[\Gamma] \to \mathbf{Z}[\mathbf{Z}/m].$$

Since the group ring $\mathbf{Z}[\mathbf{Z}/m] \cong \mathbf{Z}[q]/(q^m - 1)$ is not an integral domain (namely, has a zero divisor), we consider the composite of $\tilde{\alpha}$ and the projection

$$\pi : \mathbf{Z}[q]/(q^m - 1) \to \mathbf{Z}[q]/(\Phi_m(q)),$$

where $\Phi_m(q)$ denotes the m-th cyclotomic polynomial. Here it should be noted that π is well-defined as a ring homomorphism, because $\Phi_m(q)$ divides $q^m - 1$ in $\mathbf{Z}[q]$. Then the assumption $h_m = 1$ implies that the commutative ring $\mathbf{Z}[q]/(\Phi_m(q)) \cong \mathbf{Z}[\zeta_m]$ is a unique factorization domain (see [2]). By abuse of notation, we denote the composite $\pi \circ \tilde{\alpha}$ via $\tilde{\alpha}$ (it just corresponds to the homomorphism $\tilde{\alpha} : \mathbf{Z}[\Gamma] \to \mathbf{Z}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ in Wada's paper [4]).

Next we extend the representation ρ to the integral group ring and denote it by $\tilde{\rho}$. Then $\tilde{\rho} \otimes \tilde{\alpha}$ defines a ring homomorphism

$$\mathbf{Z}[\Gamma] \to M(n, O_K),$$

where $M(n, O_K)$ is the matrix algebra of degree n over O_K. We suppose that the group Γ has the presentation

$$\Gamma = \langle x_1, \ldots, x_u \mid r_1, \ldots, r_v \rangle.$$

Let F_u be the free group on generators $x_1, \ldots, x_u$ and define a ring homomorphism $\Psi : \mathbf{Z}[F_u] \to M(n, O_K)$ to be the composite of the surjection $\mathbf{Z}[F_u] \to \mathbf{Z}[\Gamma]$ induced by the presentation and $\tilde{\rho} \otimes \tilde{\alpha}$.

Now let us consider the $v \times u$ matrix M whose (i, j) component is the $n \times n$ matrix

$$\Psi\Big(\frac{\partial r_i}{\partial x_j}\Big) \in M(n, O_K),$$

where $\partial/\partial x$ denotes the free differential calculus (see [1]). For $1 \le j \le u$, let us denote by M_j the $v \times (u-1)$ matrix obtained from M by removing the j-th column. We now regard M_j as a $vn \times (u-1)n$ matrix with coefficients in O_K. For a $(u-1)n$-tuple of indices

$$I = (i_1, \ldots, i_{(u-1)n}) \quad (1 \le i_1 \le \cdots \le i_{(u-1)n} \le vn),$$

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we denote by M_j^I the $(u-1)n \times (u-1)n$ square matrix consisting of the i_kth rows of the matrix M_j, where $k = 1, \ldots, (u-1)n$.

Lemma 2. *In O_K, $\det M_j^I \det \Psi(1-x_k) = \pm \det M_k^I \det \Psi(1-x_j)$ holds for $1 \le j < k \le u$ and for any choice of the indices I.*

Proof. We can apply the proof of [4] Lemma 3 for our ring homomorphism Ψ. □

Hereafter we assume the following condition (C):

(C) There exists an index j so that $\det \Psi(1-x_j) \neq 0$ in O_K.

We denote by $Q_j \in O_K$ the greatest common divisor of det M_j^I for all the choices of the indices I. The algebraic integer Q_j is well-defined up to a factor of $\varepsilon \in O_K^\times$. We also define Q_j to be zero if $v < u-1$ and one if Γ is a cyclic group (i.e. $u = 1$).

Under the assumption (C), we can define the twisted invariant of Γ associated to the representation ρ to be the algebraic number

$$\Theta_{\Gamma,\rho} = \frac{Q_j}{\det \Psi(1-x_j)} \in K.$$

This is of course well-defined up to a factor of $\varepsilon \in O_K^\times$.

In order to prove Theorem 1, we need to show the invariance of $\Theta_{\Gamma,\rho}$ under the Tietze transformations (see [3]). However we can again apply Wada's argument for our situation, so that we omit the routine proof here (see [4] Theorem 1). Further we see that the twisted invariant does not depend on the choice of the basis for the representation space.

3. Examples. A few examples show what the twisted invariant is like. Our first example is a finite cyclic group $\Gamma = \mathbf{Z}/m = \langle q \mid q^m \rangle$ such that $h_m = 1$. The abelianization is the identity map; $\alpha = \mathrm{id} : \Gamma \to \langle q \mid q^m \rangle$. Every linear representation

$$\rho : \Gamma \to GL(n, \mathbf{Z})$$

is determined by the image $A = \rho(q) \in GL(n, \mathbf{Z})$ of the generator of Γ. If the representation ρ satisfies the condition (C), then we obtain

$$\Theta_{\Gamma,\rho} = \frac{1}{\det(I - qA)}.$$

A not so simple example is the following. Consider a group Γ given by

$$\Gamma = \langle x, y \mid xyx = yxy,\ (xyx)^4 = 1 \rangle.$$

The group Γ is isomorphic to the modular group $SL(2, \mathbf{Z})$. It is also known as the mapping class group of the 2-dimensional torus. As the associated homomorphism we take the abelianization $\alpha : \Gamma \to \langle q \mid q^{12} \rangle$. We then see that $h_{12} = 1$ (cf. [2]) and the cyclotomic polynomial is $\Phi_{12}(q) = q^4 - q^2 + 1$. Here we shall make a calculation in $\mathbf{Z}[q]/(q^4 - q^2 + 1)$ rather than $\mathbf{Z}[\zeta_{12}]$.

Let us write

$$r_1 = xyx - yxy \quad \text{and} \quad r_2 = (xyx)^4 - 1 = (xy)^6 - 1.$$

The free derivatives of relations r_1 and r_2 by the generator x are

$$\frac{\partial r_1}{\partial x} = 1 - y + xy \quad \text{and}$$
$$\frac{\partial r_2}{\partial x} = 1 + xy + (xy)^2 + (xy)^3 + (xy)^4 + (xy)^5.$$

First let us consider the trivial 1-dimensional representation $\mathbf{1}$ over $\mathbf{Z}$. Because $\det \Psi(1-y) = 1 - q$ (namely, the condition (C) is satisfied) and $\Psi(\partial r_2/\partial x) = 0$ in $\mathbf{Z}[q]/(q^4 - q^2 + 1)$, we can conclude

$$\Theta_{\Gamma,\mathbf{1}} = \frac{1 - q + q^2}{1 - q} \sim 1 - q + q^2,$$

where we have used a notation $\sim$ to present a relation between associated elements in the integral domain.

Next we investigate the 2-dimensional representation of Γ given by

$$\rho(x) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Direct computation shows that

$$\begin{aligned} {}^tM_2 &= \left({}^t\Psi\left(\frac{\partial r_1}{\partial x}\right) \ {}^t\Psi\left(\frac{\partial r_2}{\partial x}\right) \right) \\ &= \begin{pmatrix} 1 - q + q^2 & -q^2 & 2 + 2q^2 & 2 - 4q^2 \\ -q + q^2 & 1 - q & -2 + 4q^2 & 4 - 2q^2 \end{pmatrix}. \end{aligned}$$

Thereby we obtain $Q_2 = 1$. Further we easily see $\det \Psi(1-y) = (1-q)^2 \neq 0$, so that the twisted invariant of Γ associated to ρ is

$$\Theta_{\Gamma,\rho} = \frac{1}{(1-q)^2} \sim 1.$$

Finally we consider the braid group B_3 of three strings. Let α be the composite of the abelianization $B_3 \to \mathbf{Z}$ and the obvious homomorphism $\mathbf{Z} \to \mathbf{Z}/12$. Since the group $B_3 = \langle x, y \mid xyx = yxy \rangle$ has a representation

$$\rho : B_3 \to SL(2, \mathbf{Z}),$$

we can define its twisted invariant $\Theta_{B_3,\rho} \in \mathbf{Q}(\zeta_{12})$. From the similar computation as above, it follows that

$$\Theta_{B_3,\rho} = 2 - 3q + 2q^2.$$

On the other hand, Wada's twisted Alexander polynomial $\Delta_{B_3,\rho}(t)$ of B_3 for the representation ρ is given by

$$\Delta_{B_3,\rho}(t) = 1 + t^2.$$

We can immediately conclude this fact from the example computed in [4] Section 4 (in fact, we have only to substitute $s = -1$ into the reduced Burau representation of B_3). Therefore the discussion above implies that

Proposition 3. *The twisted invariant $\Theta_{\Gamma,\rho}$ is not a simple reduction of the twisted Alexander polynomial $\Delta_{\Gamma,\rho}(t)$.*

Remark 4. If the group Γ has a presentation of deficiency 1 and the homomorphism $\alpha : \Gamma \to \mathbf{Z}/m$ factors through $\mathbf{Z}$, then our twisted invariant for the 1-dimensional trivial representation coincides with the specialization of the original Alexander polynomial at a primitive m-th root of unity.

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