Essential self-adjointness of Dirac operators with a variable mass term

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Abstract: In this paper we study the essential self-adjointness of Dirac operators with a variable mass term $m(x)$ and an electric potential $V(x)$. We are mainly interested in the local singularities of $m(x)$ and $V(x)$. We can treat singularities of $m(x)$ and $V(x)$ which are stronger than those of Coulomb potentials.

Key words: Essential self–adjointness; Dirac operator; self–adjoint operator; singular potential.

In this note we consider the essential self-adjointness of the Dirac operator

$$H := \sum_{j=1}^{3} \alpha_j D_j + m(x)\,\beta + V(x)\,I_4$$
$$\left(x \in \mathbf{R}^3, \quad D_j = -i\frac{\partial}{\partial x_j} \right)$$

defined on $\mathbf{D} := [C_0^\infty(\mathbf{R}^3 \setminus \{0\}]^4$ in the Hilbert space $\mathbf{H} := [L^2(\mathbf{R}^3)]^4$, where

$$\alpha_j = \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} \quad (1 \le j \le 3),$$
$$\beta = \begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & -I_2 \end{pmatrix}, \quad I_4 = \begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & I_2 \end{pmatrix},$$
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$m(x)$, $V(x)$ are real valued functions and $C_0^\infty(\Omega)$ denotes the set of all C^∞-functions with compact support in Ω. We are interested mainly in the case that $m(x)$ and $V(x)$ have singularities at the origin.

The bound-state problem for $m(r) = e/r$, $V(r) = e'/r$ was studied by Vasconcelos [8], who also gave a short history of the Dirac operator with a variable mass term as a quark model. The spectral properties of H with m dominating V or vice versa at infinity as well as the case $m(x) = V(x) \to \infty$ ($|x| \to \infty$) were investigated in [7] and [10], which also contain additional references to the physical literature.

We remark that, if the real-valued functions $m(x)$ and $V(x)$ belong to $L^2_{\mathrm{loc}}(\mathbf{R}^3 \setminus \{0\})$, then the symmetric operator H has at least one self-adjoint extension. Indeed, the symmetric operator H is real with respect to the conjugation J defined by $Ju := \alpha_1\,\alpha_3\,\bar{u}$.

We summarize some notations used here.

$$\Omega := \mathbf{R}^3 \setminus \{0\}, \qquad \mathbf{R}_+ := (0, +\infty)$$
$$\mathbf{D} := [C_0^\infty(\Omega)]^4. \qquad \alpha \cdot D := \sum_{j=1}^{3} \alpha_j\,D_j,$$
$$\alpha_r := \sum_{j=1}^{3} \frac{x_j}{r}\,\alpha_j, \qquad \sigma_r := \sum_{j=1}^{3} \frac{x_j}{r}\,\sigma_j.$$

If $V = V(r)$, $m = m(r)$ are spherically symmetric, the problem of the essential self-adjointness reduces to the problem whether every one-dimensional Dirac operator L_k ($k \in \mathbf{Z} \setminus \{0\}$) in $\mathbf{R}_+$

$$L_k := \begin{pmatrix} m(r) + V(r) & -(d/dr) + (k/r) \\ (d/dr) + (k/r) & -m(r) + V(r) \end{pmatrix}$$

is of limit point type at 0, or not. If $V(r) \equiv 0$, any L_k is of limit point type at 0 for a relatively large class of $m(r)$. For example, the following proposition can be shown by Arnold–Kalf–Schneider [2], where more general theorems are given.

Proposition 1. *Let $m = m(r)$ be a real-valued function and belong to $L^1_{\mathrm{loc}}(\mathbf{R}_+)$. Then the one-dimensional Dirac operator L_k with $V = 0$ is of limit point type at 0, if $m(r)$ satisfies one of the following conditions*

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(i) $\lim_{r\to\infty} rm(r)$ *exists and is finite,*

or

(ii) $\lim_{r\to 0} r|m(r)| = \infty$, *and* $\operatorname{sgn} m$ *is constant near the origin.*

The following theorem shows a result for spherically symmetric functions $m(r)$ and scalar potentials $V(x)$ with a singularity at the origin, which is weaker than that of $m(r)$. Let $[W^{1,2}(\mathbf{R}^3)]^4$ be the Sobolev space (see, e.g., Yosida [11], ChapterI-9).

Theorem 1. *Let $m = m(r)$ be spherically symmetric and absolutely continuous in $\mathbf{R}_+$ with the derivative $m'(r) \in L^2_{\mathrm{loc}}(\mathbf{R}_+)$. Assume that $\alpha\cdot D+m\beta$ on $\mathbf{D}$ is essentially self-adjoint, and $V(x) \in L^2_{\mathrm{loc}}(\Omega)$ satisfies*

$$(1) \quad (1+\varepsilon)\left[V^2(x)+\frac{1}{4r^2}\right]+\left|m'(r)+\frac{m(r)}{r}\right| \le m^2(r)+\frac{1}{r^2} \quad (x\in\Omega)$$

for some $\varepsilon > 0$. Then H is essentially self-adjoint. If $m(r)$ is of Coulomb type, $m(r) = e/r$ $(e \in \mathbf{R})$, then the domain $D(\bar{H})$ of the closure $\bar{H}$ coincides with the Sobolev space $[W^{1,2}(\mathbf{R}^3)]^4$.

The above Theorem 1 can be proved along the line of [4] and Schmincke [6] by using the following proposition.

Proposition 2. *Assume the condition* (1). *Let $s \in \mathbf{R}$ and*

$$A := \alpha\cdot D + m\beta + \frac{i}{2r}\alpha_r + is, \quad B := V - \frac{i}{2r}\alpha_r.$$

Then we have

$$A^*A \ge \frac{1}{r^2} + m^2 - \left(m' + \frac{m}{r}\right) i\alpha_r\beta$$
$$\ge \frac{1}{r^2} + m^2 - \left|m' + \frac{m}{r}\right| \ge (1+\varepsilon)B^*B.$$

Remark 1. In Theorem 1 we can treat singularities of $m(x)$ and $V(x)$ which are stronger than those of Coulomb potential. For example, if we assume

$$(2) \quad m(r) = \frac{C_1}{r^\mu}, \quad |V(x)| \le \frac{C_2}{r^\mu}, \quad C_1 > C_2 > 0$$

and $\mu > 1$, then H is essentially self-adjoint. Indeed, if we set

$$\widetilde{m} = \frac{C_1}{r^\mu} + C$$

for $C > 0$, then V and $\widetilde{m}$ satisfy condition (1) if $\varepsilon > 0$ is sufficiently small and $C > 0$ is sufficiently large. Thus the essential self-adjointness is valid for $\alpha\cdot D + m\beta + VI_4 + C\beta$, and, therefore, for $\alpha\cdot D + m\beta + VI_4$.

In the special case $m(r) = C_1/r^\mu$ and $V(r) = C_2/r^\mu$ $(C_1 > C_2 > 0,\ \ \mu > 1)$ it follows from Theorem 3 in [2] that the equation $L_k v = 0$ has, for any $k \in \mathbf{Z}\setminus\{0\}$, a fundamental system of solutions $v_\pm$ with

$$|v_\pm(r)| = \exp\left\{\left[\pm\sqrt{C_1^2 - C_2^2}/(\mu-1) + o(1)\right] r^{1-\mu}\right\}$$

as $r \to 0$. This singularity of m therefore has the same effect on the solutions as an anomalous magnetic moment in Behncke [3].

If $\mu = 1$ in (2), then (1) is satisfied if a condition like $C_2^2 < C_1^2 + (3/4)$ holds. This result corresponds to the case $b_1 = 0$ and $s = 1/2$ in Theorem 3.1 in Arai [1]. In this case the essential self-adjointness still holds when $C_2^2 = C_1^2 + (3/4)$, if we ignore the domain property of the closure (Yamada [9]).

The following theorem states a proposition without the assumption on the spherical symmetry of $m(x)$.

Theorem 2. *Let $m,\ V \in L^2_{\mathrm{loc}}(\Omega)$ be real-valued,*

$$q(r) := \sup_{|x|=r}\left[V^2(x) + m^2(x) + 2|m(x)|\sqrt{V^2(x)+\frac{1}{4r^2}}\right]^{1/2}$$

and

$$(3) \qquad a := \sup_{r>0}\left(\frac{1}{r}\int_0^r t^2 q^2(t)\,dt\right)^{1/2} < \frac{\sqrt{3}}{2}.$$

Then H is essentially self-adjoint with $D(\bar{H}) = [W^{1,2}(\mathbf{R}^3)]^4$.

Outline of the proof. The proof is given along the line of [4]. Let $a > 0$, $\varepsilon > 0$, $s \in \mathbf{R}$ and

$$f(r) := \frac{1-\varepsilon}{2a^2r^2}\int_0^r t^2q^2(t)\,dt + \frac{\varepsilon}{4r}$$
$$A := \alpha\cdot D + if(r)\alpha_r + is,\ B := m\beta + V - if(r)\alpha_r.$$

Then we have

$$B^*B = f^2 + m^2 + V^2 + 2m[V\beta + f(i\alpha_r\beta)]$$
$$\le f^2 + m^2 + V^2 + 2|m|\sqrt{V^2(x) + \frac{1}{4r^2}}.$$

The same estimate as in [4] yields, by means of (3), that

$$A^*A - (1+\varepsilon)B^*B \ge 0.$$

for a sufficiently small $\varepsilon > 0$, which gives our assertion. □

Remark 2. Theorem 1 and Theorem 2 are directly extended to the n-dimensional Dirac operator $(n \geq 2)$

$$H := \sum_{j=1}^{n} \alpha_j D_j + m(x)\alpha_{n+1} + V(x) I_N$$
$$\big(N = 2^{[(n+1)/2]}\big),$$

where α_j $(j = 1, 2, \dots, n+1)$ are $N \times N$ Hermitian symmetric matrices satisfying $\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I_N$. Then we have only to replace (1) in Theorem 1 by

$$(1+\varepsilon)\left[V^2(x) + \frac{1}{4r^2}\right] + \left|m'(r) + \frac{m(r)}{r}\right| \leq m^2(r) + \left(\frac{n-1}{2}\right)^2 \frac{1}{r^2},$$

and (3) in Theorem 2 by

$$a := \sup_{r>0} \left(\frac{1}{r^{n-2}} \int_0^r t^{n-1} q^2(t) dt\right)^{1/2} < \frac{\sqrt{n}}{2}$$

$(n \geq 3)$.

In the plane, $H = \sigma_1 D_1 + \sigma_2 D_2 + m\sigma_3 + V I_2$ with $m(r) = C_1/r$ and $V(r) = C_2/r$ is essentially self-adjoint if and only if $|C_2| \leq |C_1|$. Indeed, L_k ($k \in \mathbf{Z} + (1/2)$ if $n = 2$) is of limit point type at 0 if and only if $C_2^2 - C_1^2 \leq k^2 - (1/4)$.

In view of Kato's inequality one has the following result on the essential self-adjointness for the case $m \equiv V$.

Theorem 3. *Assume that* $m(x) \equiv V(x)$ *is an* $L^2_{\mathrm{loc}}(\mathbf{R}^3)$ *function. Then* $H = (\alpha \cdot D) + V\beta + V$ *on* $\mathbf{D}$ *is essentially self-adjoint.*

Outline of the proof. We have only to see that the ranges of $(H \pm i)$ are dense in $\mathcal{H}$. If otherwise, we could take non-zero vectors $v := {}^t(v_1, v_2)$ and $w := {}^t(w_1, w_2) \in [L^2(\mathbf{R}^3)]^2$ such that

$$(4) \qquad \begin{cases} (\sigma \cdot D)w + 2Vv = \eta v \\ (\sigma \cdot D)v = \eta w, \end{cases}$$

where $\eta = i$ or $-i$, and

$$(5) \qquad -\Delta v + 2\eta V v = -v$$

in the sense of distributions. Then we have $Vv \in L^1_{\mathrm{loc}}(\mathbf{R}^3)$, by means of the assumption, and $\Delta v \in L^1_{\mathrm{loc}}(\mathbf{R}^3)$ by (5). Therefore, we obtain from (5) and Kato's inequality that

$$\Delta(|v_1| + |v_2|) \geq \mathrm{Re}[\mathrm{sgn}\, \bar{v}_1 \cdot \Delta v_1 + \mathrm{sgn}\, \bar{v}_2 \cdot \Delta v_2] = |v_1| + |v_2|,$$

which implies $|v_1| + |v_2| = 0$ by the same argument as in Kato [5]. Hence we have also $w = 0$ by (4), which is a contradiction. □

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