# On certain real quadratic fields with class number one

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**Abstract:** In this paper, new five real quadratic fields with norm of fundamental unit +1 and class number one are obtained.

**Key words:** Class number; real quadratic field; fundamental unit.

Throughout this paper, we denote by $\mathbf{N}$ the set of positive rational integers, and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. $\mathbf{Z}$ will mean as usual the set of rational integers. For a square-free $D \in \mathbf{N}$, the real quadratic field $\mathbf{Q}(\sqrt{D})$ will be denoted by $k$, its class number by $h_k$ and its fundamental unit $> 1$ by $\varepsilon_D = (t + u\sqrt{D})/2$. The norm map from $k$ to $\mathbf{Q}$ will be denoted by $N$.

The class number one problem requires to determine the set of all $D$ for which $h_k = 1$ under certain conditions. Let $p$ be prime congruent to 1 mod 4 and $\varepsilon_p = (u_p + t_p\sqrt{p})/2 > 1$ be the fundamental unit of the real quadratic field $\mathbf{Q}(\sqrt{p})$. In [6] Yokoi showed that there exist exactly 30 real quadratic fields $\mathbf{Q}(\sqrt{p})$ of class number one satisfying $\varepsilon_p < 2p$ with one more possible exception of prime discriminant $p$. In [2] Katayama-Katayama showed that there exist at most 44 real quadratic fields $\mathbf{Q}(\sqrt{p})$ with class number one for $1 \leq u_p \leq 300$. In [4] Mollin-Williams solved (except possibly one value) class number one problem for the more general extended Richaud-Degert (i.e. with $D = m^2 + r$ where $4m \equiv 0 \pmod{r}$) and in [5] they gave a complete generalized form of Yokoi's $p$-invariants for arbitrary real quadratic field $\mathbf{Q}(\sqrt{D})$ and all $\mathbf{Q}(\sqrt{D})$ having class number one with $n_D \neq 0$ ($n_D$ is defined in [5]).

In this paper, using the same way as in [1], we shall show that there are new five real quadratic fields with class number one for the case $N\varepsilon_D = 1$, $1 \leq u \leq 100$.

The letters $\mathbf{N}$, $\mathbf{N}_0$, $D$, $\varepsilon_D$, $t$, $u$ will always keep the meanings explained above and $n \in \mathbf{N}_0$.

**Theorem.** *With the above notations, there exist new five real quadratic fields $\mathbf{Q}(\sqrt{D})$ with class number one for $1 \leq u \leq 100$, where $D$ are those in Table with one possible exception.*

*Proof.* Using a similar way as in Prop. 1 in [1], one can find a real number $\upsilon(u)$ such that $h_k > 1$ for $n \geq \upsilon(u)$. In fact, we may take $\upsilon(u) \geq \sqrt{4 + u^2 e^{c(u)}}/u^2$. Moreover, we can choose $c(u) < 14.7$ for $1 \leq u \leq 100$. By the help of computer we obtain $\upsilon(u) = 1557/u$.

Let $q$ be an odd prime with $(D/q) = 1$. If $h_k = 1$, then we can obtain $q \geq n$ in a similar way as in the proof of Prop. 2 in [1].

In the case $h_k = 1$, it is also known that if $q_1$, $q_2$ of distinct prime factors of $D$ such that $q_2 \equiv 3 \pmod{4}$ then $D$ satisfies one of the following conditions:

i) $D = q_1$, ii) $D = q_1 q_2$, iii) $D = 2q_2$.

By the help of a computer and using Kida's UBASIC 86, we can list up the Table of the five $D$ satisfying the above necessary conditions with $h_k = 1$.

**Table**

| $u$ | $D$ |
|---|---|
| 40 | 57 |
| 77 | 893 |
| 78 | 19 |
| 84 | 22 |
| 85 | 1397 |

□

**Remark.** The real quadratic fields with class number one which are defined by Mollin and Williams in [5] can be obtained with the above theorem too.

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