Spectra of categories

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1. Introduction. We define the "Laplacian" or the "adjacency matrix" of a category $\mathcal{C}$ via

$$\Delta(\mathcal{C}) = \left({}^{\#}\mathrm{Hom}_{\mathcal{C}}(X,Y)\right)_{X,Y\in \mathrm{Ob}(\mathcal{C})}$$

where $\mathrm{Ob}(\mathcal{C})$ is the "set" (or "class") of objects, and $^{\#}$ denotes the cardinality. This notion is borrowed from the graph theory (cf. Biggs [1]), since a category is a certain "oriented graph" satisfying the associative law for edges (morphisms).

We are especially interested in the most basic case where $\mathcal{C}$ is consisting of abelian groups or modules. For convenience, when we are treating the category $\mathcal{C}$ consisting of finite abelian groups $A_1,\ldots,A_n$, we denote the Laplacian $\Delta(\mathcal{C})$ concretely as

$$\Delta(A_1,\ldots,A_n) = \left({}^{\#}\mathrm{Hom}(A_i,A_j)\right)$$

where $i,j = 1,\ldots,n$. More generally, for (left) R-modules $M_1,\ldots,M_n$ over a ring R, we simply write the associated Laplacian as

$$\Delta_R(M_1,\ldots,M_n) = \left({}^{\#}\mathrm{Hom}_R(M_i,M_j)\right).$$

Naturally $\Delta(A_1,\ldots,A_n) = \Delta_{\mathbf{Z}}(A_1,\ldots,A_n)$.

We hope to study the spectra (eigenvalues) $\mathrm{Spect}\Delta(\mathcal{C})$ of $\Delta(\mathcal{C})$. In general we expect that $\Delta(\mathcal{C})$ behaves like the classical Laplacian appearing in the differential geometry. In particular, $\Delta(\mathcal{C})$ would be symmetric and semi-positive, and the spectra would be distributed as usual.

Here we restrict ourselves to the case of $\Delta(A_1,\ldots,A_n)$ and $\Delta_R(M_1,\ldots,M_n)$ as well as their behavior as $n\to\infty$. Main results are as follows. First:

Theorem 1. *For finite abelian groups $A_1,\ldots,A_n$, $\Delta(A_1,\ldots,A_n)$ is a symmetric matrix.*

We conjecture that $\Delta(A_1,\ldots,A_n)$ is semi-positive. (The case $n=2$ is proved in [3].) The next result gives an affirmative answer for $\Delta(\mathbf{F}_p^{m_1},\ldots,\mathbf{F}_p^{m_n})$ where p is a prime.

Theorem 2. *Let $\mathbf{F}_q$ be a finite field of q elements. Then*

$$\Delta_{\mathbf{F}_q}(\mathbf{F}_q^{m_1},\ldots,\mathbf{F}_q^{m_n}) = (q^{m_i m_j})$$

is a semi-positive matrix for integers $m_i \geq 0$.

Finally we examine the behavior of spectra as $n\to\infty$ in a simple situation.

Theorem 3. *Let p_n be the n-th prime. Then the spectra $\lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \cdots \leq \lambda_n^{(n)}$ of $\Delta(\mathbf{Z}/p_1\mathbf{Z},\ldots,\mathbf{Z}/p_n\mathbf{Z})$ are all simple and located as*

$$p_1 - 1 < \lambda_1^{(n)} < p_2 - 1 < \lambda_2^{(n)} < \cdots < p_n - 1 < \lambda_n^{(n)}.$$

In particular, $\Delta(\mathbf{Z}/p_1\mathbf{Z},\ldots,\mathbf{Z}/p_n\mathbf{Z})$ is a positive matrix. Moreover, for each fixed $m\geq 1$, we have

$$\lim_{n\to\infty}\lambda_m^{(n)} = p_m - 1.$$

We remark that the convergence is very slow. For example $\lim_{n\to\infty}\lambda_1^{(n)} = 1$, but $\lambda_1^{(100)} = 1.25467\cdots$, $\lambda_1^{(1600)} = 1.23294\cdots$, and roughly

$$\lambda_1^{(n)} \approx 1 + \frac{1}{\log\log n}$$

as analyzed later.

It is well-known that spectra of Laplacians explain zeros and poles of zeta functions for Riemannian manifolds and graphs. Relations to categorical zeta functions in the direction of [2] will be treated at another occasion.

2. Symmetry. We prove Theorem 1. It is sufficient to prove the following

Lemma 1. *Let A and B be finite abelian groups, then*

$$^{\#}\mathrm{Hom}(A,B) = {}^{\#}\mathrm{Hom}(B,A).$$

Proof. Let $\hat{A}=\mathrm{Hom}(A,\mathbf{Q}/\mathbf{Z})$, $\hat{B}=\mathrm{Hom}(B,\mathbf{Q}/\mathbf{Z})$ be the dual abelian groups. (We describe abelian groups additively.) There is a natural homomorphism

$$\begin{array}{cccc} \varphi: & \mathrm{Hom}(A,B) & \longrightarrow & \mathrm{Hom}(\hat{B},\hat{A}) \\ & \Psi & & \Psi \\ & f & \longmapsto & \varphi(f) \end{array}$$

defined via

$$\varphi(f)(\chi) = \chi\circ f \qquad \text{for } \chi\in\hat{B}.$$

This φ is an injection. In fact, suppose $f \neq 0$, then there is an element $a \in A$ satisfying $f(a) \neq 0$. Then it is well-known that there exists a character $\chi \in \hat{B}$ such that $\chi(f(a)) \neq 0$. Hence $\varphi(f) \neq 0$. Thus φ is injective, so

$$^{\#}\mathrm{Hom}(A, B) \leq {}^{\#}\mathrm{Hom}(\hat{B}, \hat{A}).$$

Similarly, by duality,

$$^{\#}\mathrm{Hom}(\hat{B}, \hat{A}) \leq {}^{\#}\mathrm{Hom}(\hat{\hat{A}}, \hat{\hat{B}}) = {}^{\#}\mathrm{Hom}(A, B).$$

Hence we have

$$^{\#}\mathrm{Hom}(A, B) = {}^{\#}\mathrm{Hom}(\hat{B}, \hat{A}).$$

Since $\hat{A} \cong A$ and $\hat{B} \cong B$ as abelian groups (non-canonically), we have

$$^{\#}\mathrm{Hom}(A, B) = {}^{\#}\mathrm{Hom}(B, A).$$

□

3. Positivity. Let us prove Theorem 2. First the fact

$$\Delta_{\mathbf{F}_q}\left(\mathbf{F}_q^{m_1}, \ldots, \mathbf{F}_q^{m_n}\right) = (q^{m_i m_j})$$

is seen from

$$^{\#}\mathrm{Hom}_{\mathbf{F}_q}\left(\mathbf{F}_q^{m_i}, \mathbf{F}_q^{m_j}\right) = {}^{\#}\mathrm{M}_{m_j, m_i}(\mathbf{F}_q) = q^{m_i m_j}.$$

To show the positivity, (by induction) it is sufficient to see that $\det \Delta_{\mathbf{F}_q}\left(\mathbf{F}_q^{m_1}, \ldots, \mathbf{F}_q^{m_n}\right) \geq 0$. If $m_1, \ldots, m_n$ are not distinct, $\det \Delta_{\mathbf{F}_q}\left(\mathbf{F}_q^{m_1}, \ldots, \mathbf{F}_q^{m_n}\right) = 0$, so we may assume that $m_1, \ldots, m_n$ are distinct, and moreover (by changing the order if necessary) that $m_1 < \cdots < m_n$. Using the Vandermonde determinant we see that the matrix $R = (q^{ij})_{i,j=0,\ldots,m_n}$ of size $m_n + 1$ is positive. Since $\Delta_{\mathbf{F}_q}\left(\mathbf{F}_q^{m_1}, \ldots, \mathbf{F}_q^{m_n}\right)$ is a submatrix of R of size n, it is positive also. □

Remark 1. The above proof can be generalized to show that the matrix $R_n = (q^{m_i m_j})_{i,j=1,\ldots,n}$ is semi-positive for real numbers $m_1, \ldots, m_n$ and a real $q > 1$. In fact, it is sufficient to show that $R_n \geq 0$ when $m_1 < \cdots < m_n$. Put $m_i' = m_i - m_1$, $R_n' = (q^{m_i' m_j'})$, and $x_i' = q^{m_1 m_i'} x_i$. Then the associated quadratic form is

$$R_n\left[\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right] = \sum_{i,j} q^{m_i m_j} x_i x_j = q^{m_1^2} R_n' \left[\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}\right].$$

So we can assume that $0 = m_1 < m_2 < \cdots < m_n$. Now, if $m_1, \ldots, m_n$ are integers, $R_n > 0$ by the proof of Theorem 2. Next, suppose that $m_1, \ldots, m_n$ are rational numbers. Taking an integer $N > 0$ making $N m_i$ $(i = 1, \ldots, n)$ integers, we see $R_n > 0$ since

$$R_n = \left((q^{1/N^2})^{(Nm_i)(Nm_j)}\right).$$

Finally, real numbers $m_1, \ldots, m_n$ are limits of rational numbers, so $R_n \geq 0$.

Remark 2. Since

$$\begin{aligned} &\det \Delta_{\mathbf{F}_q}\left(\mathbf{F}_q^x, \mathbf{F}_q^y, 0\right) \\ &= \det \begin{pmatrix} q^{x^2} & q^{xy} & 1 \\ q^{xy} & q^{y^2} & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ &= q^{x^2+y^2} + 2q^{xy} - q^{x^2} - q^{y^2} - q^{2xy} \\ &= (q^{x^2} - 1)(q^{y^2} - 1) - (q^{xy} - 1)^2, \end{aligned}$$

we see the equivalence

$$\begin{aligned} &\det \Delta_{\mathbf{F}_q}\left(\mathbf{F}_q^x, \mathbf{F}_q^y, 0\right) \geq 0 \\ &\iff (q^{x^2} - 1)(q^{y^2} - 1) \geq (q^{xy} - 1)^2. \end{aligned}$$

This reminds us of the so-called q-analogue. Let $q > 1$. For a real number x, the "q-analogue" $[x]_q$ of x is defined as

$$[x]_q = \frac{q^x - 1}{q - 1}.$$

(Recall that we recover the usual situation via $q \downarrow 1$: $\lim_{q \downarrow 1}[x]_q = x$.) In this notation, from Theorem 2 and Remark 1 we have

$$(*) \qquad [x^2]_q [y^2]_q \geq [xy]_q^2$$

for all real numbers $x, y \geq 0$.

Moreover, the "q-Cauchy inequality"

$$\begin{aligned} (**) \qquad &[x_1^2 + \cdots + x_n^2]_q [y_1^2 + \cdots + y_n^2]_q \\ &\geq [x_1 y_1 + \cdots + x_n y_n]_q^2 \end{aligned}$$

holds for real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$ and real $q > 1$. These are directly proved as follows.

Proof of $(*)$. Put $u = x/\sqrt{\log q}$, $v = y/\sqrt{\log q}$. Then $(*)$ is equivalent to

$$(e^{x^2} - 1)(e^{y^2} - 1) \geq (e^{xy} - 1)^2.$$

This inequality is seen from

$$\begin{aligned} &(e^{x^2} - 1)(e^{y^2} - 1) - (e^{xy} - 1)^2 \\ &= \left(\sum_{i=1}^{\infty} \frac{x^{2i}}{i!}\right)\left(\sum_{j=1}^{\infty} \frac{y^{2j}}{j!}\right) - \left(\sum_{i=1}^{\infty} \frac{(xy)^i}{i!}\right)\left(\sum_{j=1}^{\infty} \frac{(xy)^j}{j!}\right) \\ &= \sum_{n=2}^{\infty} \frac{1}{n!}\left\{\sum_{\substack{i+j=n \\ i,j \geq 1}} \binom{n}{i}\left(x^{2i} y^{2j} - (xy)^n\right)\right\} \end{aligned}$$

$$= \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n!}\left\{\sum_{\substack{i+j=n\\ i,j\geq 1}}\binom{n}{i}\left(x^{2i}y^{2j}+x^{2j}y^{2i}-2(xy)^n\right)\right\}$$

$$= \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n!}\left\{\sum_{\substack{i+j=n\\ i,j\geq 1}}\binom{n}{i}\left(x^{i}y^{j}-x^{j}y^{i}\right)^2\right\}$$

$$\geq 0$$

□

Proof of (∗∗). Put $x=\sqrt{x_1^2+\cdots+x_n^2}$, $y=\sqrt{y_1^2+\cdots+y_n^2}$. Then, by (∗) :

$$[x_1^2+\cdots+x_n^2]_q[y_1^2+\cdots+y_n^2]_q=[x^2]_q[y^2]_q\geq[xy]_q^2.$$

The usual Cauchy inequality gives

$$xy\geq|x_1y_1+\cdots+x_ny_n|,$$

so it is easy to see

$$[xy]_q^2\geq[x_1y_1+\cdots+x_ny_n]_q^2.$$

(Notice : $q^a-1\geq\left|q^b-1\right|$ for real a,b satisfying $a\geq|b|$, since $\left|q^b-1\right|=\left|\sum_{n=1}^{\infty}\frac{b^n(\log q)^n}{n!}\right|\leq\sum_{n=1}^{\infty}\frac{|b|^n(\log q)^n}{n!}\leq\sum_{n=1}^{\infty}\frac{a^n(\log q)^n}{n!}=q^a-1$.) □

4. Spectra. We prove Theorem 3 in a slightly generalized form :

Lemma 2. *Let $1\leq a_1<a_2<\cdots\uparrow\infty$, and assume that $\sum_{n=1}^{\infty}a_n^{-1}=\infty$. Put*

$$R_n=\begin{pmatrix} a_1 & 1 & \cdots & 1\\ 1 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 1\\ 1 & \cdots & 1 & a_n\end{pmatrix}.$$

Then the spectra

$$\mathrm{Spect}R_n=\{\lambda_1^{(n)},\ldots,\lambda_n^{(n)}\}$$

are simple and located as

$$a_1-1<\lambda_1^{(n)}<a_2-1<\lambda_2^{(n)}<\cdots<a_n-1<\lambda_n^{(n)}.$$

Moreover, for each fixed $m\geq 1$, we have

$$\lim_{n\to\infty}\lambda_m^{(n)}=a_m-1.$$

Proof. Let $f_n(x)=\sum_{i=1}^{n}(x-a_i+1)^{-1}$. We first show inductively that the characteristic function is

$$\det(x-R_n)=\prod_{i=1}^{n}(x-a_i+1)\{1-f_n(x)\}$$
$$=\prod_{i=1}^{n}(x-a_i+1)-\sum_{j=1}^{n}\prod_{i\neq j}(x-a_i+1).$$

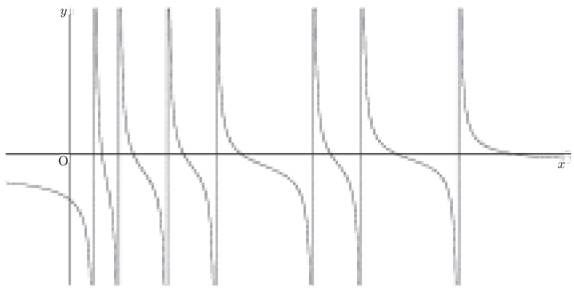

Fig. 1. The graph of $y=f_7(x)-1$.

In fact :

$$\det(x-R_n)$$
$$=\det\begin{pmatrix} x-a_1 & -1 & -1 & \cdots & -1\\ -1 & x-a_2 & -1 & \cdots & -1\\ -1 & -1 & x-a_3 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & -1\\ -1 & -1 & \cdots & -1 & x-a_n\end{pmatrix}$$
$$=\det\begin{pmatrix} x-a_1 & -1 & -1 & \cdots & -1 & a_1-1-x\\ -1 & x-a_2 & -1 & \cdots & -1 & 0\\ -1 & -1 & x-a_3 & \ddots & \vdots & \vdots\\ \vdots & \vdots & \ddots & \ddots & -1 & 0\\ -1 & -1 & \cdots & -1 & x-a_{n-1} & 0\\ -1 & -1 & \cdots & -1 & -1 & x-a_n+1\end{pmatrix}$$
$$=(x-a_n+1)\det(x-R_{n-1})+(-1)^n(x-a_1+1)$$
$$\times\det\begin{pmatrix} -1 & x-a_2 & -1 & -1 & \cdots & -1\\ -1 & -1 & x-a_3 & -1 & \cdots & -1\\ -1 & -1 & -1 & \ddots & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & \ddots & -1\\ -1 & -1 & \cdots & -1 & -1 & x-a_{n-1}\\ -1 & -1 & \cdots & -1 & -1 & -1\end{pmatrix}$$
$$=(x-a_n+1)\det(x-R_{n-1})+(-1)^n(x-a_1+1)$$
$$\times\det\begin{pmatrix} 0 & x-a_2+1 & 0 & 0 & \cdots & 0\\ 0 & 0 & x-a_3+1 & 0 & \cdots & 0\\ 0 & 0 & 0 & \ddots & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & 0 & x-a_{n-1}+1\\ -1 & -1 & \cdots & -1 & -1 & -1\end{pmatrix}$$
$$=(x-a_n+1)\det(x-R_{n-1})$$
$$-(x-a_1+1)\cdots(x-a_{n-1}+1).$$

Thus, $\lambda_1^{(n)}<\cdots<\lambda_n^{(n)}$ are coming from the solutions of $f_n(\lambda_m^{(n)})=1$, and looking at the graph of $y=f_n(x)-1$ (see Fig. 1 for $n=7$) we see that

$$a_1-1<\lambda_1^{(n)}<a_2-1<\lambda_2^{(n)}<\cdots<a_n-1<\lambda_n^{(n)}.$$

To see the behavior as $n\to\infty$, fix an $m\geq 1$ and take $n\geq m$. Then

$$\frac{1}{\lambda_m^{(n)}-(a_m-1)}$$

$$= 1 + \sum_{k=m+1}^{n} \frac{1}{a_k - 1 - \lambda_m^{(n)}} - \sum_{i=1}^{m-1} \frac{1}{\lambda_m^{(n)} - (a_i - 1)}$$

$$> 1 + \sum_{k=m+1}^{n} \frac{1}{a_k} - \sum_{i=1}^{m-1} \frac{1}{a_m - a_i}.$$

So we see that

$$0 < \lambda_m^{(n)} - (a_m - 1) < \left(1 + \sum_{k=m+1}^{n} \frac{1}{a_k} - \sum_{i=1}^{m-1} \frac{1}{a_m - a_i}\right)^{-1}.$$

Hence, letting $n \to \infty$ and using $\sum_{n=1}^{\infty} a_n^{-1} = \infty$, we have

$$\lim_{n\to\infty} \left(\lambda_m^{(n)} - (a_m - 1)\right) = 0.$$

□

Proof of Theorem 3. This follows from Lemma 2, since

$$\Delta\,(\mathbf{Z}/p_1\mathbf{Z}, \ldots, \mathbf{Z}/p_n\mathbf{Z}) = \begin{pmatrix} p_1 & 1 & \cdots & 1 \\ 1 & p_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & p_n \end{pmatrix}$$

and $\sum_{n=1}^{\infty} p_n^{-1} = \infty$ by Euler (1737). □

Remark 3. Rough estimation in Lemma 2 shows that

$$\lambda_m^{(n)} - (a_m - 1) = \left(1 + \sum_{k=m+1}^{n} \frac{1}{a_k - 1 - \lambda_m^{(n)}} - \sum_{i=1}^{m-1} \frac{1}{\lambda_m^{(n)} - a_i + 1}\right)^{-1} \approx \left(\sum_{k=m+1}^{n} \frac{1}{a_k}\right)^{-1}.$$

Hence if $a_n = p_n$ then

$$\lambda_m^{(n)} - (a_m - 1) \approx \frac{1}{\log\log n}$$

since $\sum_{k=1}^{n} p_k^{-1} \sim \log\log n$.

Numerical data :

$\lambda_1^{(10)} = 1.29262\cdots$ $\quad\lambda_1^{(20)} = 1.27739\cdots$
$\lambda_1^{(100)} = 1.25467\cdots$ $\quad\lambda_1^{(200)} = 1.24788\cdots$
$\lambda_1^{(400)} = 1.24216\cdots$ $\quad\lambda_1^{(800)} = 1.23724\cdots$
$\lambda_1^{(1600)} = 1.23294\cdots$.

Theorem 3 can be generalized as follows.

Theorem 3*. *Let $1 \le k_1 < k_2 < k_3 < \cdots$ be coprime integers. Then*

$$\mathrm{Spect}\Delta\,(\mathbf{Z}/k_1\mathbf{Z}, \ldots, \mathbf{Z}/k_n\mathbf{Z}) = \{\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}\}$$

are all simple and located as

$$k_1 - 1 < \lambda_1^{(n)} < k_2 - 1 < \lambda_2^{(n)} < \cdots < k_n - 1 < \lambda_n^{(n)}.$$

Moreover, for each fixed $m \ge 1$, $\lim_{n\to\infty} \lambda_m^{(n)}$ exists and

(a) $\lim_{n\to\infty} \lambda_m^{(n)} = k_m - 1$ *if* $\sum_{n=1}^{\infty} \frac{1}{k_n} = +\infty$,

(b) $\lim_{n\to\infty} \lambda_m^{(n)} > k_m - 1$ *if* $\sum_{n=1}^{\infty} \frac{1}{k_n} < +\infty$.

Proof. The case (a) is treated by the same method as Theorem 3 using Lemma 2. To see the case (b), we modify Lemma 2 under the condition $\sum_{n=1}^{\infty} a_n^{-1} < \infty$, where the result is the strict inequalities

$$a_m - 1 < \lim_{n\to\infty} \lambda_m^{(n)} < a_{m+1} - 1.$$

The convergence comes from $\lambda_m^{(n+1)} \le \lambda_m^{(n)}$ for $n \ge m$: notice that

$$f_{n+1}(\lambda_m^{(n)}) = 1 + \frac{1}{\lambda_m^{(n)} - (a_{n+1} - 1)} < 1.$$

□

Example for Theorem 3* :

$$\Delta(\mathbf{Z}/p_1^2\mathbf{Z}, \ldots, \mathbf{Z}/p_n^2\mathbf{Z}) = \begin{pmatrix} p_1^2 & 1 & 1 & \cdots & 1 \\ 1 & p_2^2 & 1 & \cdots & 1 \\ 1 & 1 & p_3^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & \cdots & 1 & p_n^2 \end{pmatrix}.$$

Then the minimum eigenvalues are

$\lambda_1^{(10)} = 3.751057\cdots$ $\quad\lambda_1^{(20)} = 3.748495\cdots$
$\lambda_1^{(100)} = 3.747255\cdots$ $\quad\lambda_1^{(200)} = 3.747172\cdots$
$\lambda_1^{(400)} = 3.747139\cdots$ $\quad\lambda_1^{(800)} = 3.747126\cdots$
$\lambda_1^{(1600)} = 3.747121\cdots$.

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