On the generalized Nörlund summability of a sequence of Fourier coefficients

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1. Introduction. Let $f(t)$ be a periodic function with period 2π on $(-\infty, \infty)$ and Lebesgue integrable over $(-\pi, \pi)$. Then the conjugate series of the Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

of f is

$$\sum_{n=1}^{\infty}(b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} B_n(t).$$

Since Fejer [3] found the relations between the "jump" of $f(t)$ at $t = x$ and the sequence $\{nB_n(x)\}$, there are many results which show how the behaviour of $f(t)$ in the neighborhood of $t = x$ controls the convergence of the sequence $\{nB_n(x)\}$ to the jump in the sense of summability. To state the most recent result of Khare and Tripathi [5], we need the following definitions.

Given two sequences $p = \{p_n\}$ and $q = \{q_n\}$, the convolution $(p * q)$ is defined by

$$(p * q)_n = \sum_{k=0}^{n} p_{n-k} q_k = \sum_{k=0}^{n} p_k q_{n-k}.$$

Let $\{s_n\}$ be a sequence. When $(p * q)_n \neq 0$ for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$ obtained by putting

$$t_n^{p,q} = \frac{1}{(p * q)_n} \sum_{k=0}^{n} p_{n-k} q_k s_k.$$

If $\lim_{n\to\infty} t_n^{p,q}$ exists and is equal to s, then the sequence $\{s_n\}$ is said to be summable (N, p_n, q_n) to the value s.

If $s_n \to s$ $(n \to \infty)$ induces $t_n^{p,q} \to s$ $(n \to \infty)$, then the method (N, p_n, q_n) is called to be regular. The necessary and sufficient condition for (N, p_n, q_n) method to be regular is $\sum_{k=0}^{n} |p_{n-k}\ q_k| = O(|(p * q)_n|)$ and $p_{n-k} = o(|(p * q)_n|)$ as $n \to \infty$ for every fixed $k \geq 0$ (see Borwein [2]).

The method (N, p_n, q_n) reduces to the Nörlund method (N, p_n) if $q_n = 1$ for all n and to the Riesz method $(\overline{N}, q_n)$ if $p_n = 1$ for all n. We know that (N, p_n) mean or $(\overline{N}, q_n)$ mean includes as a special case Cesàro and harmonic means or logarithmic mean, respectively.

The method $(N, p_n, q_n)(C, 1)$ is obtained by superimposing the method (N, p_n, q_n) on the Cesàro mean (C,1) of order one (see Astrachan [1]).

Throughout this paper, we shall use the following notations:

$$\psi_x(t) = \{f(x + t) + f(x - t) - l\},$$
$$\Psi_x(t) = \int_0^t |\psi_x(u)|du,$$

for any fixed x $(-\infty < x < \infty)$ and a constant l depending on x. For two sequence $\{p_n\}$ and $\{q_n\}$, we define $P(t)$ $(0 \leq t < \infty)$ and R_n $(n = 0, 1, 2, \ldots)$ by

$$P(t) = \sum_{k=0}^{[t]} p_k \quad \text{and} \quad R_n = (p * q)_n = \sum_{k=0}^{n} p_{n-k} q_k,$$

where $[t]$ denotes the integral part of t.

Theorem KT (Khare and Tripathi [5]). *Let (N, p_n, q_n) be regular Nörlund method defined by a non-negative, non-increasing sequence $\{p_n\}$ and a non-negative, non-decreasing sequence $\{q_n\}$. If the condition*

$$\text{(1.1)} \quad \int_{\pi/n}^{\delta} \frac{|\psi_x(t)|}{t} P\left(\frac{\pi}{t}\right) dt = o(R_n q_n^{-1}) \quad (n \to \infty)$$

holds for a number δ, $0 < \delta < \pi$, then the sequence $\{nB_n(x)\}$ is summable $(N, p_n, q_n)(C, 1)$ to l/π.

In this paper, by generalizing a result of Hirokawa and Kayashima [4], we shall give a theorem which contains Theorem KT.

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2. Statement of our result. We define another function $R_n(t)$ $(0 \le t < n+1)$ with a non-negative integer n by

$$R_n(t) = \int_0^t r_n(u)du,$$

where $r_n(u) = p_k q_{n-k}$ for $k \le u < k+1$ $(k = 0, 1, 2, \dots, n)$.

Theorem. *Let (N, p_n, q_n) be a regular Nörlund method defined by a non-negative, non-increasing sequence $\{p_n\}$ and a non-negative, non-decreasing sequence $\{q_n\}$.*

If the condition

$$(2.1)\quad \int_{\pi/n}^{\delta} \Psi_x(t) \left| \frac{d}{dt} \frac{R_n(\pi/t)}{t} \right| dt = o(R_n) \ \text{as } n \to \infty$$

holds for a number δ, $0 < \delta < \pi$, then the sequence $\{nB_n(x)\}$ is summable $(N, p_n, q_n)(C, 1)$ to l/π.

Now we shall show that Theorem contains Theorem KT. First of all, we remark

$$\frac{d}{dt} \frac{R_n(\pi/t)}{t} < 0$$

for any t, $0 < \pi/t < n+1$, because

$$(2.2)\quad \frac{d}{dt} \frac{R_n(\pi/t)}{t} = \frac{(-\pi/t) r_n(\pi/t) - R_n(\pi/t)}{t^2}.$$

Since the condition (1.1) implies $\Psi_x(t) = o(t)$ $(t \to 0)$, we have

$$\begin{aligned}
&\int_{\pi/n}^{\delta} \Psi_x(t) \left| \frac{d}{dt} \frac{R_n(\pi/t)}{t} \right| dt \\
&= \left[-\Psi_x(t) \frac{R_n(\pi/t)}{t} \right]_{\pi/n}^{\delta} + \int_{\pi/n}^{\delta} \frac{|\psi_x(t)|}{t} R_n\left(\frac{\pi}{t}\right) dt \\
&= o\Big(R_n(n) - R_n\left(\frac{\pi}{\delta}\right)\Big) \\
&\quad + o\bigg(q_n \int_{\pi/n}^{\delta} \frac{|\psi_x(t)|}{t} P\left(\frac{\pi}{t}\right) dt\bigg) = o(R_n),
\end{aligned}$$

by virtue of the fact that $R_n(\pi/t) \le q_n P(\pi/t)$. Hence the conditions (1.1) is implied in the condition (2.1).

From Theorem we immediately obtain the following corollary with another conditions (2.3) and (2.4) which are considered by Singh [9] with respect to convergence of conjugate series.

Corollary. *Let (N, p_n, q_n) be a regular Nörlund method defined by a non-negative, non-increasing sequence $\{p_n\}$ and a non-negative, non-decreasing sequence $\{q_n\}$ such that*

$$(2.3)\quad q_n \int_1^n \frac{\lambda(u)}{u} du = O(R_n) \quad \text{as} \quad n \to \infty,$$

where $\lambda(u)$ is a positive function of u. If the condition

$$(2.4)\quad \int_0^t |\psi_x(u)| du = o\left(\frac{t\lambda(\pi/t)}{P(\pi/t)}\right) \quad \text{as} \quad t \to 0$$

holds, then the sequence $\{nB_n(x)\}$ is summable $(N, p_n, q_n)(C, 1)$ to l/π.

Proof. Since the sequence $\{p_k q_{n-k}\}_{k=0}^n$ is a non-increasing sequence of k, we have

$$0 \le \frac{\pi}{t} r_n\left(\frac{\pi}{t}\right) \le R_n\left(\frac{\pi}{t}\right) \quad \left(0 < \frac{\pi}{t} < n+1\right).$$

Hence we obtain

$$\frac{d}{dt} \frac{R_n(\pi/t)}{t} = O\left(\frac{R_n(\pi/t)}{t^2}\right),$$

from (2.2). If the conditions (2.3) and (2.4) hold, then we have

$$\begin{aligned}
&\int_{\pi/n}^{\delta} \Psi_x(t) \left| \frac{d}{dt} \frac{R_n(\pi/t)}{t} \right| dt \\
&= o\left(\int_{\pi/n}^{\delta} \frac{t\lambda(\pi/t)}{P(\pi/t)} \cdot \frac{R_n(\pi/t)}{t^2} dt\right) \\
&= o\left(\int_{\pi/n}^{\delta} \frac{\lambda(\pi/t) R_n(\pi/t)}{tP(\pi/t)} dt\right) \\
&= o\left(\int_{\pi/\delta}^{n} \frac{\lambda(y) R_n(y)}{yP(y)} dy\right) \\
&= o\left(\sum_{k=2}^{n} \int_{k-1}^{k} \frac{\lambda(y) R_n(y)}{yP(y)} dy\right) \\
&= o\left(\sum_{k=2}^{n} \frac{R_n(k)}{P(k-1)} \int_{k-1}^{k} \frac{\lambda(y)}{y} dy\right) \\
&= o\left(q_n \sum_{k=2}^{n} \frac{P_{k-1}}{P(k-1)} \int_{k-1}^{k} \frac{\lambda(y)}{y} dy\right) \\
&= o\left(q_n \int_1^n \frac{\lambda(y)}{y} dy\right) = o(R_n),
\end{aligned}$$

which is the condition (2.1). □

3. Proof of Theorem. We need the following lemma for the proof of Theorem.

Lemma. *Let $\{p_n\}$ be a non-negative, non-increasing sequence and $\{q_n\}$ be a non-negative, non-decreasing sequence. If we put*

$$K_n(t) = \frac{1}{R_n} \sum_{k=1}^{n} p_{n-k}\, q_k \left\{ \frac{\sin kt}{kt^2} - \frac{\cos kt}{t} \right\},$$

then we have

$$K_n(t) = O(n) \quad \left(0 < t \le \frac{\pi}{n}\right) \tag{3.1}$$

and

$$K_n(t) = O\left(\frac{R_n(\pi/t)}{tR_n}\right) \quad \left(\frac{\pi}{n} < t \le \pi\right). \tag{3.2}$$

Proof. If $0 < t \le \pi/n$, then we have

$$\begin{aligned} K_n(t) &= O\left(\frac{1}{R_n}\sum_{k=1}^{n} p_{n-k}q_k(k^2 t)\right) \\ &= O\left(\frac{n}{R_n}\sum_{k=1}^{n} p_{n-k}q_k\right) = O(n), \end{aligned}$$

which is (3.1).

To prove (3.2), we first put

$$\begin{aligned} I_1 &= \frac{1}{R_n}\sum_{k=1}^{n} p_{n-k}q_k\frac{\sin kt}{kt^2}, \\ I_2 &= \frac{1}{R_n}\sum_{k=1}^{n} p_{n-k}q_k\frac{\cos kt}{t}. \end{aligned}$$

Then

$$|K_n(t)| \le |I_1| + |I_2|. \tag{3.3}$$

By virtue of the fact that

$$\sum_{k=1}^{n} p_{n-k}q_k \cos kt = O\left(R_n\left(\frac{\pi}{t}\right)\right) \quad \left(\frac{\pi}{n} < t \le \pi\right)$$

(see [6]), we immediately obtain

$$|I_2| = O\left(\frac{R_n(\pi/t)}{tR_n}\right) \tag{3.4}$$

for any t $(\pi/n < t \le \pi)$. Next, by dividing I_1 into two parts I_{11} and I_{12}:

$$\begin{aligned} I_{11} &= \frac{1}{tR_n}\sum_{k=0}^{\tau-1} p_k q_{n-k}\frac{\sin(n-k)t}{(n-k)t}, \\ I_{12} &= \frac{1}{tR_n}\sum_{k=\tau}^{n-1} p_k q_{n-k}\frac{\sin(n-k)t}{(n-k)t}, \end{aligned}$$

where $\tau = [\pi/t]$, we have

$$|I_1| \le |I_{11}| + |I_{12}|. \tag{3.5}$$

If $\pi/n < t \le \pi$, then

$$I_{11} = O\left(\frac{R_n(\pi/t)}{tR_n}\right) \tag{3.6}$$

By Abel's transformation, we have

$$\begin{aligned} I_{12} &= O\biggl(\frac{1}{t^2R_n}\biggl\{\sum_{k=\tau}^{n-2}(p_k q_{n-k} - p_{k+1}q_{n-k-1}) \\ &\qquad + p_{n-1}q_1 + p_\tau q_{n-\tau}\biggr\}\biggr) \\ &= O\left(\frac{p_\tau q_{n-\tau}}{t^2R_n}\right) = O\left(\frac{R_n(\pi/t)}{tR_n}\right), \end{aligned}$$

because

$$\left|\sum_{k=1}^{n} \sin\frac{kt}{k}\right| \le \frac{\pi}{2} + 1$$

for any positive integer n (see [10]). Thus (3.2) evidently follows from (3.3)-(3.7). □

Proof of Theorem. From the method of Mohanty and Nanda [7], we have

$$\begin{aligned} &\frac{1}{n}\sum_{k=1}^{n} kB_k(x) - \frac{l}{\pi} \\ &= \frac{1}{\pi}\int_0^\delta \psi_x(t)\left\{\frac{\sin nt}{nt^2} - \frac{\cos nt}{t}\right\}dt + o(1). \end{aligned}$$

by Riemann-Lebesgue's theorem. Since the method (N, p_n, q_n) is regular, in order to prove Theorem, it is sufficient to show that

$$\begin{aligned} &\frac{1}{R_n}\sum_{k=1}^{n} p_{n-k}q_k\frac{1}{\pi}\int_0^\delta \psi_x(t)\left\{\frac{\sin kt}{kt^2} - \frac{\cos kt}{t}\right\}dt \\ &= \frac{1}{\pi}\int_0^\delta \psi_x(t)K_n(t)dt = o(1). \end{aligned}$$

Now we write

$$\begin{aligned} \frac{1}{\pi}\int_0^\delta \psi_x(t)K_n(t)dt &= \frac{1}{\pi}\biggl(\int_0^{\pi/n} + \int_{\pi/n}^\delta\biggr)\psi_x(t)K_n(t)dt \\ &= I_3 + I_4, \end{aligned}$$

say. If the condition (2.1) is satisfied, we obtain

$$\Psi_x(t) = o(t) \quad (t \to 0)$$

from (3.1) (see [8]). Hence we have

$$I_3 = O(n)\biggl(\int_0^{\pi/n} |\psi_x(t)|dt\biggr) = o(1).$$

Next, we obtain by (3.2)

$$\begin{aligned} I_4 &= O\biggl(\frac{1}{R_n}\int_{\pi/n}^\delta |\psi_x(t)|\frac{R_n(\pi/t)}{t}dt\biggr) \\ &= O\biggl(\frac{1}{R_n}\left[\Psi_x(t)\frac{R_n(\pi/t)}{t}\right]_{\pi/n}^\delta\biggr) \\ &\quad + O\biggl(\frac{1}{R_n}\int_{\pi/n}^\delta \Psi_x(t)\left|\frac{d}{dt}\frac{R_n(\pi/t)}{t}\right|dt\biggr) \\ &= o\biggl(\frac{R_n(\pi/\delta)}{R_n} + \frac{R_n(n)}{R_n}\biggr) + o(1) = o(1) \ (n \to \infty). \end{aligned}$$

Therefore, these complete the proof of Theorem. □

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