On the Green function of the p-Laplace equation for Riemannian manifolds

By Takeshi KURA

Department of Mathematics, Faculty of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, Hiroshima 739-8526

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1. Introduction. Let M be a smooth noncompact connected complete Riemannian n-manifold without boundary and $1 < p \leq n$ be a constant. The purpose of this note is to give geometric criteria for the existence and nonexistence of the p-Green function of the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \tag{1}$$

on M. In case $p = 2$, Ichihara [5] has given geometric criteria for the existence and nonexistence of the 2-Green function of M. Kasue [6] has given estimates of 2-Green function of M. We shall extend the results of Ichihara [5] and Kasue [6] to (1). The p-Green function of (1) for M was first defined by Holopainen [2]. We refer to Holopainen [3], [4] and Tanaka [8] for the p-Green function of M and related topics.

For open set G in M, a function $u \in locW^{1,p}(G) \bigcap C(G)$ is said to be p-harmonic in G if u satisfies

$$\int_G \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle dv_M = 0, \qquad \text{for all } \varphi \in C_0^\infty(G),$$

where $\langle , \rangle$ and v_M are the Riemannian metric and volume measure of M respectively. Let q be a fixed point in M. For a bounded smooth domain G containing q, a function $g = g(\cdot, q)$ is said to be p-Green function of G with pole q if it satisfies the following conditions:

$$\begin{aligned} &g \text{ is } p\text{-harmonic in } G \setminus \{q\},\\ &\lim_{x\to y} g(x) = 0 \qquad \text{for every y} \in \partial G,\\ &\lim_{x\to q} g(x) = \infty,\\ &-\operatorname{div}(|\nabla g|^{p-2}\nabla g) = \delta_q \text{ in } G, \end{aligned}$$

in the sense of distributions, i.e.

$$\int_G \langle |\nabla g|^{p-2}\nabla g, \nabla\varphi\rangle dv_M = \varphi(q), \quad \text{for all } \varphi \in C_0^\infty(G).$$

Let $\{G_l\}_{l\in\mathbf{N}}$ be an exhaustion of M by bounded smooth domains G_l such that $q \in G_1, G_l \subset G_{l+1}$, and $M = \bigcup_l G_l$. Holopainen [2] proved that there exists a p-Green function g_l of G_l such that the sequence $\{g_l\}$ is increasing. By the Harnack's convergence theorem ([1, Theorem 6.14]) $g = \lim_{l\to\infty} g_l$ is either p-harmonic in $M \setminus \{q\}$ or identically $+\infty$ in M. In the former case g is said to be a p-Green function of M. The uniqueness of the p-Green function of M is not known except $p = n$. In case $p = n$, Holopainen [2] proved the uniqueness.

2. Results. Let SM be the unit tangent bundle of M. For a $v \in S_xM$, we set $\alpha_v(t) = \exp(tv)$ and $N(v) = \{w \in S_xM | \langle v, w\rangle = 0\}$. Set $h(x) = d(q, x)$ in M where d is the Riemannian distance of M. Suppose that Ω is a bounded open set in M containing q and put $\Omega_1 = \Omega \setminus \{q\}$. Let $\mathcal{S}$ be the set of the positive p-harmonic functions of (1) in Ω_1 with isolated singularity at q and

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \delta_q \text{ in } \Omega.$$

Let

$$G_p(t) = \begin{cases} t^{\frac{n-p}{p-1}}, & if\ 1 < p < n,\\ -\log t, & if\ p = n. \end{cases}$$

The following lemma is due to Serrin [7].

Lemma 1. *There exist two positive constants* c_1, c_2 *such that*

$$c_1 \leq \liminf_{x\to q} \frac{u(x)}{G_p(h(x))} \leq \limsup_{x\to q} \frac{u(x)}{G_p(h(x))} \leq c_2, \quad \textit{for } u \in S$$

Let $\mathrm{K_M}$ and $\mathrm{Ric_M}$ denote the sectional and Ricci curvatures of M respectively. Put $k_p = (n-p)/(p-1)$ if $1 < p < n$ and $k_n = 1$. Let $R : [0, \infty) \to \mathbf{R}$ be a continuous function. We assume that the injectivity radius of q is infinity and that the initial value problem

$$\begin{cases} f'' + R(t)f = 0 & \text{in } (0, \infty),\\ f(0) = 0, \quad f'(0) = 1, \end{cases} \tag{2}$$

has the positive solution f. We have by Lemma 1 the following theorems:

Theorem 1. *Suppose that R satisfies*

$$\mathrm{K_M}(\alpha_v'(t), w) \leq R(t)$$

for $v \in S_qM,\ 0 < t < \infty,\ w \in N(\alpha_v'(t))$. *Let*

$$\int_T^\infty f(t)^{\frac{1-n}{p-1}} dt < \infty \quad \text{for some } T > 0.$$

Then the p-Green function g of M satisfies

$$g(x, q) \leq k_p c_2 \int_{h(x)}^\infty f(t)^{\frac{1-n}{p-1}} dt \quad \text{in } M.$$

In particular, M has a p-Green function.

Theorem 2. *Suppose that R satisfies*

$$\mathrm{Ric_M}(\alpha_v'(t), \alpha_v'(t)) \geq (n-1)R(t)$$

for $v \in S_qM,\ 0 < t < \infty$. *Then the p-Green function g of M satisfies*

$$g(x, q) \geq k_p c_1 \int_{h(x)}^\infty f(t)^{\frac{1-n}{p-1}} dt \quad \text{in } M.$$

In particular, if

$$\int_T^\infty f(t)^{\frac{1-n}{p-1}} dt = \infty, \quad \text{for some } T > 0,$$

then M has no p-Green function.

Let $B(t)$ be the geodesic ball of radius t about q. If $F(t)$ is a C^2 function on $(0, \infty)$ satisfying $F'(t) < 0$, then $u(x) = F(h(x))$ satisfies

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = -|F'(h)|^{p-2}(|F'(h)|\Delta h - (p-1)F''(h))$$

in $M \setminus \{q\}$. Let f be the positive solution of (2). Then

$$k_p^{-1} = \lim_{t \to 0} \frac{1}{G_p(t)} \int_t^\infty f(s)^{\frac{1-n}{p-1}} ds.$$

Proof of Theorem 1. By the Hessian comparison theorem(cf. Kasue [6, Lemma 2.18]), we have

$$\Delta h(x) \geq (n-1)\frac{f'(h(x))}{f(h(x))} \quad \text{for} \quad x \in M \setminus \{q\}.$$

Let

$$F(t) = \int_t^\infty f(s)^{\frac{1-n}{p-1}} ds, \quad u(x) = F(h(x)).$$

Then

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) \leq 0 \quad \text{in} \quad M \setminus \{q\}.$$

Let G be a bounbed smooth domain in M such that $\Omega \subset G$ and g_1 be a p-Green function of G. Then $g_1 \in \mathcal{S}$. Fix a small $\varepsilon > 0$. By Lemma 1, there exists $\delta > 0$ such that $g_1(x) \leq (k_p^{-1} - \varepsilon)^{-1}(c_2 + \varepsilon)u(x)$ in $B(\delta) \setminus \{q\}$. The comparison principle([1, Theorem 7.6]) implies that $g_1(x) \leq (k_p^{-1} - \varepsilon)^{-1}(c_2 + \varepsilon)u(x)$ in $G \setminus \{q\}$. By letting $\varepsilon \to 0$ we obtain $g_1(x) \leq k_p c_2 u(x)$ in $G \setminus \{q\}$, and the theorem is proved. □

Proof of Theorem 2. By the Laplacian comparison theorem(cf. Kasue [6, Lemma 2.5]), we have

$$\Delta h(x) \leq (n-1)\frac{f'(h(x))}{f(h(x))} \quad \text{for} \quad x \in M \setminus \{q\}.$$

There exists $T_1 > 0$ such that $\Omega \subset B(T_1)$. Fix $T > T_1$. Choose a bounded smooth domain G in M such that $B(T) \subset G$. Let g_1 be a Green function of G. Then $g_1 \in \mathcal{S}$. Set

$$F(t) = \int_t^T f(s)^{\frac{1-n}{p-1}} ds, \quad u(x) = F(h(x)).$$

Then

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) \geq 0 \quad \text{in} \quad B(T) \setminus \{q\}.$$

Fix a small $\varepsilon > 0$. By Lemma 1, there exists $\delta > 0$ such that $g_1(x) \geq (k_p^{-1} + \varepsilon)^{-1}(c_2 - \varepsilon)u(x)$ in $B(\delta) \setminus \{q\}$. The comparison principle([1, Theorem 7.6]) implies that $g_1(x) \geq (k_p^{-1} + \varepsilon)^{-1}(c_1 - \varepsilon)u(x)$ in $B(T) \setminus \{q\}$. By letting $\varepsilon \to 0$ we obtain $g_1(x) \geq k_p c_1 u(x)$ in $B(T) \setminus \{q\}$. If g is a p-Green function of M, then $g(x) \geq k_p c_1 u(x)$ in $B(T) \setminus \{q\}$. The theorem follows by letting $T \to \infty$. □

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