

A note on q -analogues of Dirichlet series

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Abstract: In this note, we study the q -Dirichlet series $Z_q(s)$ which was evaluated at non-positive integers by Satoh. We consider the values of $Z_q(s)$ at positive integers. By letting $q \rightarrow 1$, we get the Euler formulas for $\zeta(2k)$ and the recent formulas for $\zeta(2k+1)$ given by Cvijović-Klinowski. We also consider the relation between $Z_q(s)$ and Jackson's q - Γ -function.

Key words: Dirichlet series; q -Analogue; q -Benoulli numbers.

1. Introduction. In [5], the modified q -Bernoulli numbers $\{\tilde{\beta}_k(q)\}$ were defined by $\tilde{\beta}_0(q) = (q - 1)/\log q$, and

$$\sum_{j=0}^k \binom{k}{j} q^j \tilde{\beta}_j(q) - \tilde{\beta}_k(q) = \begin{cases} 1 & (k = 1) \\ 0 & (k \geq 2) \end{cases},$$

for an indeterminate q . Recently Satoh considered the q -series $Z_q(s) = \sum_{n \geq 1} q^n/[n]^s$ which interpolated $\{\tilde{\beta}_k(q)\}$ at *non-positive* integers for a complex number q with $|q| < 1$, where $[x] = (1 - q^x)/(1 - q)$ (see [4] §4).

In this note, we consider the values of $Z_q(s)$ at *positive* integers. Assume that q is a complex number with $|q| < 1$ and let $\tilde{Z}_q(s) = Z_q(s) + (1 - q)^s/((1 - s)\log q)$. Note that if $q \rightarrow 1$ then $\tilde{Z}_q(s) \rightarrow \zeta(s)$ for any complex number s with $Re(s) > 1$, where $\zeta(s)$ is the Riemann zeta function. We prove the following series representations for $\tilde{Z}_q(s)$ at positive integers by the same method as that in [6] §1. Note that an empty sum is to be interpreted as nil.

Theorem. For a positive integer k ,

$$\begin{aligned} (1) & (2k - 1)\pi \tilde{Z}_q(2k) - 2k \sum_{n=1}^{\infty} \frac{q^n \sin([n]\pi)}{[n]^{2k+1}} \\ & + \pi \sum_{n=1}^{\infty} \frac{q^n \cos([n]\pi)}{[n]^{2k}} \\ & = \sum_{j=1}^{k-1} \frac{(-1)^{j-1} \pi^{2j+1}}{(2j+1)!} (2k-2j-1) \tilde{Z}_q(2k-2j) \\ & - \frac{(1-q)^{2k+1}}{\log q} \sin\left(\frac{\pi}{1-q}\right) \\ & + (-1)^{k+1} \pi^{2k+1} \left\{ \frac{\tilde{\beta}_1(q) + 1}{(2k+1)!} \right. \end{aligned}$$

$$\begin{aligned} & \left. + \sum_{m=1}^{\infty} \frac{(-1)^m \pi^{2m}}{(2k+2m+1)!} \tilde{\beta}_{2m+1}(q) \right\}; \\ (2) & 2k \tilde{Z}_q(2k+1) - 2k \sum_{n=1}^{\infty} \frac{q^n \cos([n]\pi)}{[n]^{2k+1}} \\ & - \pi \sum_{n=1}^{\infty} \frac{q^n \sin([n]\pi)}{[n]^{2k}} \\ & = \sum_{j=1}^{k-1} \frac{(-1)^{j-1} \pi^{2j}}{(2j)!} (2k-2j) \tilde{Z}_q(2k-2j+1) \\ & - \frac{(1-q)^{2k+1}}{\log q} \cos\left(\frac{\pi}{1-q}\right) \\ & + (-1)^{k+1} \pi^{2k} \sum_{m=0}^{\infty} \frac{(-1)^m \pi^{2m}}{(2k+2m)!} \tilde{\beta}_{2m}(q). \end{aligned}$$

We can let $q \rightarrow 1$ in above relations by the same consideration as that in [6] §2. So we get the following.

Corollary. For a positive integer k ,

$$\begin{aligned} (1) & \zeta(2k) = \frac{(-1)^k (2\pi)^{2k}}{2\{2^{2k}(k-1) + 1\}} \\ & \times \left[\sum_{j=1}^{k-1} \frac{(-1)^{j-1} (2j-1) \zeta(2j)}{(2k-2j+1)! \pi^{2j}} \right. \\ & \left. - \frac{1}{2(2k+1)!} \right]; \\ (2) & \zeta(2k+1) = \frac{(-1)^k (2\pi)^{2k}}{k(2^{2k+1} - 1)} \\ & \times \left[\sum_{j=1}^{k-1} \frac{(-1)^{j-1} j \zeta(2j+1)}{(2k-2j)! \pi^{2j}} \right. \\ & \left. + \sum_{m=0}^{\infty} \frac{(2m)! \zeta(2m)}{(2m+2k)! 2^{2m}} \right]. \end{aligned}$$

The above formula (1) is an analogue of that in

[6, Theorem A]. From this, we can inductively obtain the Euler formulas $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, \dots . The above formula (2) is equal to the recent result given by Cvijović-Klinowski (see [3, Theorem A]), which is generalization of the Euler formula for $\zeta(3)$ (c.f. [2]). They proved their result by using the functional equation and the summation formula for $\zeta(s)$. But our result is established only by elementary calculations of uniformly convergent series.

Lastly we define the q -series $\tilde{Z}_q(s, b)$ which satisfies that $\lim_{q \rightarrow 1} \tilde{Z}_q(s, b)$ coincides with the Hurwitz zeta function $\zeta(s, b)$. And we consider the relation between $\tilde{Z}_q(s, b)$ and Jackson's q - Γ -function $\Gamma_q(s)$.

2. Proof of Theorem. Satoh proved that

$$\tilde{Z}_q(1 - k) = \begin{cases} -\frac{\tilde{\beta}_k(q)}{k} & (\text{if } k \geq 2) \\ -\tilde{\beta}_1(q) - 1 & (\text{if } k = 1) \end{cases}$$

(see [4]§4, Example 1). A short proof of this fact will be given in §4 below. In fact Satoh proved more general result by using the formal groups. By the above result, we can verify the relations in Theorem by excuting direct calculation as in the same method as that in [6] §1 as follows. Since $|q| < 1$, we have

$$\begin{aligned} & 2k \sum_{n=1}^{\infty} \frac{q^n \sin([n]\theta)}{[n]^{2k+1}} - \theta \sum_{n=1}^{\infty} \frac{q^n \cos([n]\theta)}{[n]^{2k}} \\ &= 2k \sum_{n=1}^{\infty} \frac{q^n}{[n]^{2k+1}} \sum_{j=0}^{\infty} \frac{(-1)^j \theta^{2j+1}}{(2j+1)!} [n]^{2j+1} \\ & - \theta \sum_{n=1}^{\infty} \frac{q^n}{[n]^{2k}} \sum_{j=0}^{\infty} \frac{(-1)^j \theta^{2j}}{(2j)!} [n]^{2j} \\ &= \sum_{j=0}^{k-1} \frac{(-1)^j \theta^{2j+1}}{(2j+1)!} \left\{ (2k-2j-1) \sum_{n=1}^{\infty} \frac{q^n}{[n]^{2k-2j}} \right\} \\ & + \sum_{j=k}^{\infty} \frac{(-1)^j \theta^{2j+1}}{(2j+1)!} \left\{ (2k-2j-1) \sum_{n=1}^{\infty} q^n [n]^{2j-2k} \right\}, \\ &= \sum_{j=0}^{k-1} \frac{(-1)^j \theta^{2j+1}}{(2j+1)!} \left\{ (2k-2j-1) \tilde{Z}_q(2k-2j) \right. \\ & \quad \left. + \frac{(1-q)^{2k-2j}}{\log q} \right\} \\ & + \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} \left(\tilde{\beta}_1(q) + 1 + \frac{1}{\log q} \right) \\ & + \sum_{j=k+1}^{\infty} \frac{(-1)^j \theta^{2j+1}}{(2j+1)!} \left(\tilde{\beta}_{2j-2k+1}(q) + \frac{(1-q)^{2k-2j}}{\log q} \right) \end{aligned}$$

$$\begin{aligned} &= \sum_{j=0}^{k-1} \frac{(-1)^j \theta^{2j+1}}{(2j+1)!} (2k-2j-1) \tilde{Z}_q(2k-2j) \\ & + \frac{(1-q)^{2k+1}}{\log q} \sin\left(\frac{\theta}{1-q}\right) \\ & + \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} (\tilde{\beta}_1(q) + 1) \\ & + \sum_{j=k+1}^{\infty} \frac{(-1)^j \theta^{2j+1}}{(2j+1)!} \tilde{\beta}_{2j-2k+1}(q). \end{aligned}$$

Thus we get the proof of Theorem (1). The formula (2) can be proved similarly.

3. Proof of Corollary. Let $F_q(t)$ be the generating function of $\{\tilde{\beta}_k(q)\}$. We can see that $F_q(t)$ is determined as a solution of the following q -difference equation (see [5]§1):

$$F_q(t) = e^t F_q(qt) - t, \quad F_q(0) = \frac{(q-1)}{\log q}.$$

We can easily verify that

$$F_q(t) = \frac{q-1}{\log q} e^{\frac{t}{1-q}} - t \sum_{n=0}^{\infty} q^n e^{[n]t}.$$

So $F_q(t)$ is holomorphic in the whole complex plane if $|q| < 1$, and $F_1(t) = t/(e^t - 1)$.

Lemma. Let r and d be real numbers with $0 < r < 2\pi$ and $0 < d < 1$. Then there exists a constant $M(r, d) > 0$ such that

$$\left| \frac{\tilde{\beta}_k(q)}{k!} \right| \leq \frac{M(r, d)}{r^k},$$

for $k \geq 0$, if $d \leq q \leq 1$.

Proof. Let C_r be a circle around O of radius r in the complex plane. By the above consideration, we can see that $F_q(t)$ is continuous as a function of (q, t) on the compact set $[d, 1] \times C_r$. So we let $M(r, d) = \text{Max}|F_q(t)|$ on $[d, 1] \times C_r$. By the fact that

$$\frac{\tilde{\beta}_k(q)}{k!} = \frac{1}{2\pi i} \int_{C_r} F_q(t) t^{-k-1} dt,$$

we get the proof of Lemma.

By using well-known Weierstrass' M-test for uniform convergence, we can let $q \rightarrow 1$ in the formulas in Theorem. Since $k \geq 1$, we can see that

$$\begin{aligned} (1-q)^{2k} \sin\left(\frac{\pi}{1-q}\right) &\rightarrow 0, \\ (1-q)^{2k} \cos\left(\frac{\pi}{1-q}\right) &\rightarrow 0, \end{aligned}$$

and $\tilde{\beta}_k(q) \rightarrow B_k$ if $q \rightarrow 1$. By using Euler's formula

$$\zeta(2k) = \frac{(-1)^{k-1} 2^{2k-1} B_{2k} \pi^{2k}}{(2k)!},$$

and the fact $B_{2k+1} = 0$ for $k \geq 1$, we get the proof of Corollary.

4. q -series $\tilde{Z}_q(s, b)$. We define the q -Bernoulli polynomials $\{\tilde{\beta}_k(x, q)\}$ by

$$G_q(t, x) = \sum_{k=0}^{\infty} \tilde{\beta}_k(x, q) \frac{t^k}{k!},$$

where $G_q(t, x) = F_q(q^x t) e^{[x]t}$. We can easily verify that

$$\tilde{\beta}_k(1, q) = \tilde{\beta}_k(q) \text{ if } k \neq 1$$

and

$$\tilde{\beta}_1(1, q) = \tilde{\beta}_1(q) + 1.$$

Since

$$\begin{aligned} G_q(t, x) &= \left(\frac{q-1}{\log q} e^{\frac{q^x t}{1-q}} - q^x t \sum_{n=0}^{\infty} q^n e^{[n]q^x t} \right) e^{[x]t} \\ &= \frac{q-1}{\log q} e^{\frac{t}{1-q}} - t \sum_{n=0}^{\infty} q^{n+x} e^{[n+x]t}, \end{aligned}$$

we have

$$\tilde{\beta}_k(x, q) = -\frac{(1-q)^{1-k}}{\log q} - k \sum_{n=0}^{\infty} q^{n+x} [n+x]^{k-1},$$

for any non-negative integer k . So we define the q -series

$$\tilde{Z}_q(s, b) = \frac{(1-q)^s}{(1-s)\log q} + \sum_{n=0}^{\infty} \frac{q^{n+b}}{[n+b]^s},$$

for a real number b with $0 < b \leq 1$. Note that $\tilde{Z}_q(s, 1) = \tilde{Z}_q(s)$, and $\tilde{Z}_q(s, b) \rightarrow \zeta(s, b)$ if $q \rightarrow 1$. We can see that

$$\tilde{Z}_q(1-k, b) = -\tilde{\beta}_k(b, q)/k$$

for any natural number k . Especially when $b = 1$, Satoh's result is given (see §2 above). Now we recall Jackson's q - Γ -function defined by

$$\Gamma_q(x) = (1-q)^{1-x} \frac{\prod_{n=0}^{\infty} (1-q^{n+1})}{\prod_{n=0}^{\infty} (1-q^{n+x})}.$$

In [1], Askey proved that

$$\frac{d^2}{dx^2} \log \Gamma_q(b) = \left(\frac{\log q}{q-1} \right)^2 \sum_{n=0}^{\infty} \frac{q^{n+b}}{[n+b]^2}.$$

So we get the following.

Proposition.

$$\frac{d^2}{dx^2} \log \Gamma_q(b) = \left(\frac{\log q}{q-1} \right)^2 \tilde{Z}_q(2, b) + \log q.$$

Since $(\log q)/(q-1) \rightarrow 1$ and $\log q \rightarrow 0$ if $q \rightarrow 1$, the above relation can be regarded as a q -analogue of the classical one $(d^2/dx^2) \log \Gamma(b) = \zeta(2, b)$.

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