A note on the Rankin-Selberg method for Siegel cusp forms of genus 2

By Taro Horie*)

Graduate School of Polymathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, Aichi 464–8602

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The purpose of this note is to give an explicit relation between certain Dirichlet series and spinor zeta functions attached to Siegel cusp forms of genus 2; a part of results in [7] is generalized to the case of *any level*. Thereby we point out that the method of [7] to study spinor zeta functions is applicable to higher levels.

1. Notations. We use standard notations, found in [2]. We let $\Gamma_2 := \mathrm{Sp}_2(\mathbf{Z})$ be integral symplectic 4×4-matrices and Γ_1 be the elliptic full modular group. We set

$$\Gamma_g(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \middle| \, C \equiv O(\mathrm{mod}\ N) \right\}.$$

where A, B, C, D are $g\times g$-matrices. We let $\Gamma_1^J(N)$ be the semi direct product of $\Gamma_1(N)$ and $\mathbf{Z}^2$, which is called the *Jacobi group of level N*.

$\mathcal{H}_g$ denotes the Siegel upper half space of genus g consisting of complex $g\times g$-matrices with positive definite imaginary part. We often write

$$Z = X + iY = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathcal{H}_2.$$

Let k be an integer > 2 and χ be a Dirichlet character modulo N. We write $S_k(N,\chi)$ for the space of holomorphic cusp forms on $\mathcal{H}_2$ of weight k and character χ with respect to $\Gamma_2(N)$, and $J_{k,l}^{\mathrm{cusp}}(N,\chi)$ for the space of holomorphic Jacobi cusp forms on $\mathcal{H}_1\times\mathbf{C}$ of weight k, character χ and index l with respect to $\Gamma_1^J(N)$. The Petersson inner product on these spaces are normalized by

$$\langle F, G\rangle_N := \int_{\Gamma_2(N)\backslash\mathcal{H}_2} F(Z)\bar{G}(Z)\,|Y|^{k-3}\,dX\,dY$$
$$(F, G \in S_k(N,\chi),\ Z = X+iY \in \mathcal{H}_2),$$
$$\langle \phi, \psi\rangle_N := \int_{\Gamma_1^J(N)\backslash\mathcal{H}_1\times\mathbf{C}} \phi(\tau,z)\,\bar{\psi}(\tau,z) \times \exp\left(-\frac{4\pi l y^2}{v}\right) v^{k-3}du\,dv\,dx\,dy$$
$$(\phi,\psi\in J_{k,l}^{\mathrm{cusp}}(N,\chi),\ \tau = u+iv\in\mathcal{H}_1,\ z = x+iy\in\mathbf{C}).$$

We write simply $\mathbf{e}(*)$ for $\exp(2\pi i *)$.

2. Statement of Result.

Definition. Let F, $G \in S_k(N,\chi)$ be Siegel cusp forms of level N and let M be a natural number which divides N. For each $\gamma\in\mathrm{Sp}_2(\mathbf{Z})$, we write

$$F|_k\gamma(Z) = \sum_{n\geq 1}\phi_{n,\gamma}(\tau,z)\mathbf{e}\left(\frac{n\tau'}{N}\right),$$
$$G|_k\gamma(Z) = \sum_{n\geq 1}\psi_{n,\gamma}(\tau,z)\mathbf{e}\left(\frac{n\tau'}{N}\right).$$

Then we define the *Rankin convolution series* $D_{F,G;M}(s)$ as $\zeta(2s-2k+4)$ times

$$(1)\ \sum_{n\geq 1}\left\{\int_{\mathcal{F}}\sum_{\gamma\in\Gamma_2(N)\backslash\Gamma_2(M)}\phi_{n,\gamma}(\tau,z)\,\bar{\psi}_{n,\gamma}(\tau,z) \times\exp\left(-\frac{4\pi n y^2}{vN}\right)v^{k-3}\,du\,dv\,dx\,dy\right\}n^{-s},$$

where $\mathcal{F}$ is a fundamental domain $\Gamma_1^J(M)\backslash\mathcal{H}_1\times\mathbf{C}$, and define its gamma factor by

$$D^*_{F,G;M}(s) := (2\pi)^{-2s}\,\Gamma(s)\,\Gamma(s-k+2)\,D_{F,G;M}(s).$$

In a special case of $M = N$, this is an obvious generalization of the symmetric square series defined by Rankin in the case of genus 1 ([10]):

$$D_{F,G;N}(s) = \frac{1}{N^s}\zeta(2s-2k+4)\sum_{n\geq 1}\frac{\langle\phi_n,\psi_n\rangle_N}{n^s},$$

where ϕ_n (resp. ψ_n) denotes the n-th Fourier-Jacobi coefficient of F (resp. G).

On the other hand, if $F(Z)\in S_k(N,\chi)$ is a Hecke eigenform with $T(n)F = \lambda_F(n)F$ for all the Hecke operators $T(n)$ with $(n,N)=1$, one can associate with F the *spinor zeta function* which is an Euler product of the form

$$(2)\ Z_F(s) := \prod_{\substack{p:\mathrm{prime}\\(p,N)=1}} Q_{F,p}(p^{-s})^{-1},$$

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$$Q_{F,p}(t) := 1 - \lambda(p)t + (\lambda_F(p)^2 - \lambda(p^2) - \chi(p^2)p^{2k-4})t^2 - \lambda_F(p)\chi(p^2)p^{2k-3}t^3 + \chi(p^4)p^{4k-6}t^4$$

for $\mathrm{Re}(s) \gg 0$ (cf. [1], (4.3.35), Proposition 3.3.35, Exercise 3.3.38 and (4.2.11)). Its natural gamma factor is defined by

$$Z_F^*(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_F(s).$$

The modular forms which play an important role in relating (1) to (2) are Poincaré series. For a negative discriminant $D = r^2 - 4n$, we define the D-th Jacobi Poincaré series of index 1 by

$$P_{D,N}(\tau,z) = P_{D,N,\chi}(\tau,z) := \sum_{\gamma \in \Gamma_{1,\infty}^J \backslash \Gamma_1^J(N)} \bar{\chi}(d) \times \frac{1}{(c\tau+d)^k}\mathbf{e}\left(-\frac{cz^2}{c\tau+d} + \lambda^2\frac{a\tau+b}{c\tau+d} + \frac{2\lambda z}{c\tau+d}\right) \times \mathbf{e}\left(n\frac{a\tau+b}{c\tau+d} + r\frac{z+\lambda(a\tau+b)}{c\tau+d}\right) \in J_{k,1}^{\mathrm{cusp}}(N,\chi),$$

where we write $\gamma = \left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \lambda, \mu\right) \in \Gamma_1^J(N)$ and $\Gamma_{1,\infty}^J := \left\{\left(\begin{pmatrix} \pm 1 & b \\ 0 & \pm 1\end{pmatrix}, 0, \mu\right)\right\} \subset \Gamma_1^J(N)$. We define a *Siegel modular form* $\mathcal{P}_{D,N}(Z)$ as the "Maass lifting" of $P_{D,N}(\tau,z)$ (see the section 3).

Now let us state our main result.

Theorem. *For a cusp form $F \in S_k(N,\chi)$ and a natural number M which divides N, we set*

$$\mathrm{Tr}_M^N(F) := \sum_{\gamma \in \Gamma_2(N)\backslash\Gamma_2(M)} F|_k\gamma(Z) \ \in S_k(M,\chi).$$

Suppose that $\mathrm{Tr}_M^N(F)$ is a non-zero Hecke eigenform for all the Hecke operators $T(n)$ with $(n,M)=1$. Then for any negative fundamental discriminant D we have a relation

$$(3) \quad D_{F,\mathcal{P}_{D,M};M}(s) = d_{\mathrm{Tr}_M^N(F),D}(s) Z_{\mathrm{Tr}_M^N(F)}(s).$$

Here for $\mathrm{Tr}_M^N(F)(Z) = \sum_{Q>0}\tilde{A}(Q)\mathbf{e}(\mathrm{tr}QZ)$, by writing the indices of Fourier coefficients by integral ideals in $\mathbf{Q}(\sqrt{D})$, we define a Dirichlet series

$$(4) \quad d_{\mathrm{Tr}_M^N(F),D}(s) := \frac{1}{N^s}\sum_{\Im | M^\infty} \tilde{A}(\Im)\mathrm{N}\Im^{-s},$$

where $\Im$ runs through all ideals of the maximal order in $\mathbf{Q}(\sqrt{D})$ such that each of the prime ideals which divides $\Im$ also divides M and $\mathrm{N}\Im$ denotes the norm of $\Im$. $d_{\mathrm{Tr}_M^N(F),D}(s)$ is also defined by a following meromorphic function on the whole s-plane :

$$\frac{1}{N^s h}\sum_{\xi}\prod_{\wp | M}\left(1 - \frac{\bar{\xi}(\wp)}{\mathrm{N}\wp^{s-k+2}}\right)^{-1}\sum_{i=1}^{h}\xi(\Im_i)\tilde{A}(\Im_i),$$

where $h = h(D)$ is the class number of $\mathbf{Q}(\sqrt{D})$, $\wp$ runs through all prime ideals dividing M, $\{\Im_i\}_{i=1,\ldots,h}$ denotes a set of representatives of the ideal class group and ξ runs through all ideal class characters.

We shall write down the special case of $M = N$. Let $F \in S_k(N,\chi)$ be a non-zero Hecke eigenform, then for any negative fundamental discriminant D we have a relation

$$\zeta(2s-2k+4)\sum_{n\geq 1}\frac{\langle \phi_n, P_{D,N}|V_n\rangle_N}{n^s} = \sum_{\Im | N^\infty}\frac{A(\Im)}{\mathrm{N}\Im^s} \times Z_F(s),$$

where $A(\Im)$ (resp. $\phi_n(\tau,z)$) denotes the $\Im$-th (resp. n-th) Fourier (resp. Fourier-Jacobi) coefficient of F, and V_n denotes the n-th Hecke operator which maps $J_{k,1}^{\mathrm{cusp}}(N,\chi)$ to $J_{k,n}^{\mathrm{cusp}}(N,\chi)$ (see the section 3).

3. Outline of proof. The proof proceeds along the lines of the second proof in [7], which uses "Andrianov's formula" (7). First we prove

Theorem-definition (Saito-Kurokawa-Maass lifting). *Let $\phi(\tau,z) \in J_{k,1}^{\mathrm{cusp}}(N,\chi)$ be a Jacobi cusp form of index 1. Then we have a lifting map from $J_{k,1}^{\mathrm{cusp}}(N,\chi)$ to $S_k(N,\chi)$ via*

$$\phi(\tau,z) \mapsto \sum_{l\geq 1}\phi|V_l(\tau,z)\mathbf{e}(l\tau'),$$

where V_l denotes the l-th Hecke operator which maps $J_{k,1}^{\mathrm{cusp}}(N,\chi)$ to $J_{k,l}^{\mathrm{cusp}}(N,\chi)$ and defined by

$$(\phi|V_l)(\tau,z) := l^{k-1}\sum_{\substack{\gamma\in\Gamma_1(N)\backslash \mathrm{M}_2(\mathbf{Z}) \\ ad-bc=l,\ c|N,\ (a,N)=1}}\chi(a) \times\frac{1}{(c\tau+d)^k}\mathbf{e}\left(\frac{-lcz^2}{c\tau+d}\right)\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{lz}{c\tau+d}\right),$$

where we write $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$. We denote the image of this map by $S_k^*(N,\chi)$ and call it the Maass space of level N and character χ.

Proof . For $N = 1$, see the proof of Theorem 6.2 in [2]. In the general case, the same proof also works (cf. [8]). □

We let $\mathcal{P}_{D,N}(Z)$ be the image of $P_{D,N}(\tau,z)$ (see the section 2) in $S_k^*(N,\chi)$ under the lifting map:

$$\mathcal{P}_{D,N}(Z) := \sum_{l\geq 1} P_{D,N}|V_l(\tau,z)\mathbf{e}(l\tau').$$

For a half integral symmetric matrix $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ with $D := b^2 - 4ac$, we can associate with T a binary quadratic form

$$Q(x,y) = [a,b,c](x,y) = ax^2 + bxy + cy^2$$

of discriminant D, and an integral ideal (of some order) in $\mathbf{Q}(\sqrt{D})$:

$$\Im = a\mathbf{Z} + \frac{-b+\sqrt{D}}{2}\mathbf{Z}.$$

We occasionary write $A(Q)$ or $A(\Im)$ for Fourier coefficients of Siegel modular forms.

Proof of theorem. Write the Fourier expansion and the Fourier-Jacobi expansion of $\mathrm{Tr}(F)$ by

$$\begin{aligned}\mathrm{Tr}_M^N F(Z) &= \sum_{Q>0} \tilde{A}(Q)\mathbf{e}(\mathrm{tr}QZ) \\ &= \sum_{l>0} \tilde{\phi}_l(\tau,z)\mathbf{e}(l\tau')\end{aligned}$$

respectively.

We note that for all $\gamma \in \Gamma_2(M)$

$$\mathcal{P}_{D,M}|_k\gamma(Z) = \sum_l P_{D,M}|V_l(\tau,z)\mathbf{e}(l\tau'),$$

hence in the notations of (1) in Definition

$$\psi_{n,\gamma} = \begin{cases} 0 & n \text{ is not divisible by } N \\ P_{D,M}|V_l & \text{if } n = Nl \end{cases}.$$

So, the Nl-th coefficient of $\zeta(2s-2k+4)^{-1} D_{F,\mathcal{P}_{D,M};M}(s)$ is equal to

$$\langle \sum_\gamma \phi_{lN,\gamma}, P_{D,M}|V_l\rangle_M = \langle \tilde{\phi}_l|V_l^*, P_{D,M}\rangle_M,$$

where $V_l^* : J_{k,l}^{\mathrm{cusp}}(M,\chi) \to J_{k,1}^{\mathrm{cusp}}(M,\chi)$ denotes the adjoint operator of V_l (note that $\sum_\gamma \phi_{lN,\gamma} = \tilde{\phi}_l$ is a Jacobi form of level M and index l).

At first, we notice an important fact that $P_{D,N}(\tau,z)$ (D-th Jacobi Poincaré series in $J_{k,1}^{\mathrm{cusp}}(N,\chi)$) is characterized by

(5) $\langle \phi, P_{D,N}\rangle_N := \lambda_{k,l,D}\ c_{n,r}(\phi)\ (\forall\phi \in J_{k,1}^{\mathrm{cusp}}(N,\chi))$,

where $\lambda_{k,l,D} := \frac{1}{2}\Gamma(k-\frac{3}{2})\pi^{-k+3/2}l^{k-2}|D|^{-k+3/2}$ and $c_{n,r}(\phi)$ denotes the (n,r)-th Fourier coefficient of ϕ with $D = r^2 - 4n$. For the proof confer [5], p.520 (In [5] only the full modular case is treared, but we can easily follow the proof in the general case).

Next, we must calculate the action of V_l^* explicitly as in [7], p.554-557. This step is the key.

Then using this calculation and the characterization (5) of $P_{D,M}$. We get

$$\begin{aligned}&\langle \tilde{\phi}_l|V_l^*, P_{D,M}\rangle_M \\ &= \sum_{i=1}^{h(D)} \sum_{d|l,(l/d,M)=1} \bar{\chi}(l/d)d^{k-2}n(Q_i;d)\tilde{A}\left(\frac{l}{d}Q_i\right),\end{aligned}$$

where $\{Q_i\}_{i=1,\dots,h(D)}$ is a set of representatives of binary quadratic forms of discriminant D and $n(Q_i;d)$ denotes

$$\sharp\{s(\mathrm{mod}\ 2d)|s^2 \equiv D\ (4d),\ [\frac{s^2-D}{4d},s,d] \sim Q_i\}.$$

Observing

$$\sum_{n\geq 1} n(Q_i;n)n^{-s} = \zeta_{Q_i}(s)\zeta(2s)^{-1},$$

where $\zeta_{Q_i}(s)$ is the (partial) zeta function of the class of Q_i, we obtain

(6) $D_{F,\mathcal{P}_{D,M};M}(s)$

$$= N^{-s}\sum_{i=1}^{h(D)} \zeta_{Q_i}(s-k+2)R_{Q_i,\mathrm{Tr}_M^N(F),M}(s),$$

with

$$R_{Q_i,\mathrm{Tr}_M^N(F),M}(s) := \sum_{n\geq 1,(n,M)=1} \frac{\bar{\chi}(n)\tilde{A}(nQ_i)}{n^s}.$$

We now recall Andrinov's formula in [1], Theorem 4.3.16. Take any fundamental discriminant D and any Hecke eigenform $F(Z) = \sum_{Q>0} A(Q)\mathbf{e}(\mathrm{tr}QZ) \in S_k(M,\chi)$. Then for any class character ξ of the class group $H(D)$ and any completely multiplicative function ω on $\mathbf{N}_{(M)} := \{n \in \mathbf{N}|(n,M) = 1\}$, it holds

(7)
$$\begin{aligned}&A_\xi(s)\prod_{\substack{\wp:\text{prime ideal}\\(\wp,M)=1}}\left(1-\frac{\chi(\wp)\xi(\mathrm{N}\wp)\omega(\mathrm{N}\wp)}{(\mathrm{N}\wp)^{s-k+2}}\right) \\ &\times \prod_{\substack{p:\text{prime}\\(p,M)=1}} Q_{F,p}(\omega(p)p^{-s})^{-1} \\ &= \sum_{i=1}^{h(D)} \xi(Q_i)\sum_{n\in\mathbf{N}_{(M)}} \frac{\omega(n)A(nQ_i)}{n^s},\end{aligned}$$

with

$$A_\xi(s) := \sum_{i=1}^{h(D)} \xi(Q_i)A(Q_i).$$

Inverting this formula for $F = \mathrm{Tr}_M^N(F)$, $\omega = \bar{\chi}$ and instituting in (6), we obtain (3). □

4. Applications. $D_{F,G;M}(s)$ defined in the section 2 has an integral representation ([6], Lemma 2):

$$D^*_{F,G;M}(s) = \pi^{-k+2}N^{-s}\langle FE^*_{s-k+2,M}, G\rangle_N,$$

where $E_{s,M}(Z)$ denotes a certain Eisenstein series of Klingen-Siegel type. From this we can deduce

Proposition ([6], Proposition 1 and the section 4)**.** *All $D_{F,G;M}(s)$'s with $M|N$ have a meromorphic continuation to* $\mathbf{C}$. *$\Pi_{f|N}(1-f^{s-k+2})D_{F,G;N}(s)$, where f runs through all square-free positive integers dividing N, is entire if $\langle F, G\rangle_N = 0$ and otherwise has a simple pole at $s = k$ as its only singularity, and if $N = p$ is a prime number we have*

$$\mathrm{Res}_{s=k}D_{F,G;p}(s) = \frac{4^k\pi^{k+2}}{(k-1)!}\frac{1}{(1+p^2)p^k}\langle F, G\rangle_p.$$

Furthermore there exists a functional equation

$$P(s)D^*_{F,G;N}(2k-2-s)$$
$$= a \textit{ finite sum of } const.n^sD^*_{F,G;M}(s),$$

where M, n are natural numbers with $M|N$ and $P(s)$ is a finite product of $1 - f^{2(k-s)}$ with $f|N$. For example, if $N = p$ is a prime number we have

$$(1-p^{2(k-s)})D^*_{F,G;p}(2k-2-s)$$
$$= (1-p^{2(s-k+2)})D^*_{F,G;p}(s)$$
$$-(1-p^{2(s-k+1)})D^*_{F,G;1}(s).$$

Using Proposition and Theorem in the case of '$M = N$' we obtain

Corollary 1. *Let $F \in S_k(N,\chi)$ be a non-zero Hecke eigen form. Suppose that $d_{F,D}(s)$ defined by (4) is not identically zero for some fundamental discriminant D. Then $Z_F(s)$ has a meromorphic continuation to the whole s-plane, the possible poles of $d_{F,D}(s)Z_F(s)$ are $s = k$ and those corresponding to zeros of $\prod_{f|N}(1-f^{s-k+2})$, where f runs through all square-free positive integers dividing N. If $N = p$ is a prime number, we have*

$$\frac{1}{\pi^{k+2}\langle F, \mathcal{P}_{p,D}\rangle_p}\mathrm{Res}_{s=k}Z_F(s) \in \mathbf{Q}(F, \zeta_{h(D)}),$$

where $\zeta_{h(D)}$ is a primitive $h(D)$-th root of unity.

Furthermore there exists a functional equation

$$P(s)d_{F,D}(2k-2-s)Z^*_F(2k-2-s)$$
$$= const.l^sd_{F,D}(s)Z^*_F(s)$$
$$+ \textit{ a finite sum of } const.n^sP_2(s)D^*_{F,\mathcal{P}_{N,D};M}(s),$$

where M, l, n are natural numbers with $M|N$ and $P(s)$ is a finite product of $1 - f^{2(k-s)}$ with $f|N$. For example, if $N = p$ is a prime number we have

$$(1-p^{2(k-s)})d_{F,D}(2k-2-s)Z^*_F(2k-2-s)$$
$$= (1-p^{2(s-k+2)})d_{F,D}(s)Z^*_F(s)$$
$$-(1-p^{2(s-k+1)})D^*_{F,\mathcal{P}_{p,D};1}(s).$$

Remark. Similar results to Corollary 1 for principal congruence subgroups are reported in [3], p.457 (without proof).

Corollary 2 (cf. [4], [7], [9])**.** *Let $F \in S_k(N, \chi)$ be a non-zero Hecke eigen form in the orthogonal compliment of $S^*_k(N,\chi)$ (the Maass space, see the section 3). Then $\prod_{f|N}(1-f^{s-k+2})d_{F,D}(s)Z_F(s)$, where f runs through all square-free positive integers dividing N, is holomorphic for all s.*

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