A Thermodynamic formalism for one dimensional cellular automata

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1. Introduction. The topological dynamics on the product space of a finite set $(S^{\mathbf{Z}}, \tau)$ is called one dimensional cellular automata (CA) if there exists a finite subset $\Lambda \subset \mathbf{Z}$ and a local map $f : S^\Lambda \mapsto S$ which satisfy $(\tau x)_i = f(x_{i+j}; j \in \Lambda)$ for all $x \in S^{\mathbf{Z}}$. Since its transient dynamics show various phenomena, the orbit structure is too much complicated to treat using typical method of the dynamical systems [2].

Let (X, σ) be the full shift over the symbol S. The dynamical zeta function with potential function $V \in BV(X)$ is defined as follows [1]:

$$\zeta(z, V) = \exp(\sum_{k>0} \frac{z^k}{k} Z_k(V)),$$
$$Z_k(V) = \sum_{x \in \mathrm{Fix}(X, \sigma^k)} e^{-S_k V(x)}$$

where $S_k V(x) = V(x) + V(\sigma x) + \cdots + V(\sigma^{k-1} x)$ and $\mathrm{Fix}(X, \sigma^k) = \{x \in X;\ \sigma^k x = x\}$.

Suppose that the potential function depends only one site x_0, i.e. $V(x) = V(x_0)$ for all $x = \{x_i\}_{i \in \mathbf{Z}}$, then the thermodynamical limit

$$P(V_n) = \lim_{k \to \infty} \frac{1}{k} \log Z_k(V_n),\ V_n(x) = V(\tau^n x)$$

exists for each n since $V_n(x)$ depends at most $np + 1$ site.

The problem is the behavior of $P(V_n)$ as $n \to \infty$ and the relation with the transient dynamics of (X, τ).

In this paper we will show the structure of $\zeta(z, V_n)$ formally by the structure matrix of the local map $f^{(n)}$, where $f^{(n)} : S^{np+1} \mapsto S$ is defined naturally from f.

2. Results. At first, we define the $\#S^{np} \times \#S^{np}$ structure matrix $M_n(a)$, $a \in S$ with the index set is $\{r_1 \cdots r_{np};\ r_i \in S\}$.

Definition 2.1. The structure matrix $M_n(a)$ is as follows:

$$(M_n(a))_{r_1 \cdots r_{np}, s_1 \cdots s_{np}} = \begin{cases} 1 & \text{if } r_2 \cdots r_{np} = s_1 \cdots s_{np-1} \\ & \text{and } f^{(n)}(r_1 \cdots r_{np} s_{np}) = a, \\ 0 & \text{otherwise.} \end{cases}$$

We can express $Z_k(V_n)$ using trace formula of $M_n(a)$ as follows:

$$\begin{aligned} Z_k(V_n) &= \sum_{x \in \mathrm{Fix}(X, \sigma^k)} e^{-S_k V(\tau^n x)} \\ &= \sum_{\substack{x \in \mathrm{Fix}(X, \sigma^k) \\ y \in \mathrm{Fix}(\tau^n X, \sigma^k) \\ \tau^n x = y}} e^{-S_k V(y)} d(y, k) \\ &\quad (d(x,k) = \#\{x \in \mathrm{Fix}(X, \sigma^k); \\ &\qquad \tau^n x = y, y \in \mathrm{Fix}(\tau^n X, \sigma^k)\}) \\ &= \sum_{y \in \mathrm{Fix}(\tau^n X, \sigma^k)} e^{-S_k V(y)} \\ &\qquad \mathrm{trace}(M_n(y_0) \cdots M_n(y_{k-1})) \\ &= \sum_{y \in \mathrm{Fix}(\tau^n X, \sigma^k)} \mathrm{trace}(e^{-V(y_0)} M_n(y_0) \\ &\qquad \cdots e^{-V(y_{k-1})} M_n(y_{k-1})) \\ &= \sum_{\substack{y_0 \in S \\ \cdots \\ y_{k-1} \in S}} \mathrm{trace}(e^{-V(y_0)} M_n(y_0) \\ &\qquad \cdots e^{-V(y_{k-1})} M_n(y_{k-1})) \\ &= \mathrm{trace}((\sum_{a \in S} e^{-V(a)} M_n(a))^k). \end{aligned}$$

Thus, the zeta function is written by the determinant form.

$$\begin{aligned} \zeta(z, V_n) &= \exp(\sum_{k>0} \frac{1}{k} \mathrm{trace}(z \sum_{a \in S} e^{-V(a)} M_n(a))^k) \\ &= \exp(\mathrm{trace}(\sum_{k>0} \frac{1}{k} (z \sum_{a \in S} e^{-V(a)} M_n(a))^k)) \\ &= \exp(\mathrm{trace}(-\log(I - z \sum_{a \in S} e^{-V(a)} M_n(a)))) \\ &= \det(I - z \sum_{a \in S} e^{-V(a)} M_n(a))^{-1} \\ &\quad (|z| < e^{-P(V_n)}). \end{aligned}$$

As a result, we have the theorem.

Theorem 2.2. *For one dimensional CA, we have the formula*:

$$\zeta(z, V_n) = \det(I - z \sum_{a \in S} e^{-V(a)} M_n(a))^{-1}.$$

3. Example. In this section, we assume that $S = \{0, 1\}$, $V(0) = 0$ and $V(1) = -\log p$ for simplify. We treat

$$D_n(z) = \zeta(z, V_n)^{-1} = \det(I - z \sum_{a \in S} e^{-V(a)} M_n(a))$$

which is the reciprocal of zeta function.

Each of the two examples shown below has only one attractive fixed point 0^∞ and V_n converges to the constant function $V(0)$ for almost every points. These are the cases that the limit $\lim_{n\to\infty} D_n(z)$ exists.

Example 3.1. We take the local map $f : S^3 \to S$

$$f(abc) = \begin{cases} 1 & \text{if } abc = 111, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$M_1(0) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ M_1(1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} &D_n(z) \\ &= 1 - (1+p)z - (1-p)z^2(1 + z + \cdots + z^{2n-1}) \\ &= 1 - (1+p)z - (1-p)z^2 \frac{1 - z^{2n}}{1 - z} \\ &= \frac{1 - (2+p)z + (1+p)z^2 - (1-p)z^2(1 - z^{2n})}{1 - z} \\ &= \frac{1 - (2+p)z + 2pz^2 + (1-p)z^{2(n+1)}}{1 - z} \\ &= \frac{(1-2z)(1-pz) + (1-p)z^{2(n+1)}}{1 - z}. \end{aligned}$$

Thus $P(V_n) = \log 2 - O(2^{-n})$.

Example 3.2. We take the local map $f : S^3 \to S$,

$$f(abc) = \begin{cases} 1 & \text{if } abc = 111, 101, 110, \\ 0 & \text{otherwise.} \end{cases}$$

Then we get

$$M_1(0) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ M_1(1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} D_{n+1}(z) &= D_n(z) + (p-1)z^3 z^n (z+1)^n \\ D_0(z) &= 1 - (1+p)z. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &D_n(z) \\ &= D_0(z) + \sum_{k=0}^{n-1} (p-1)z^3 z^k (z+1)^k \\ &= 1 - (1+p)z + (p-1)\frac{1 - z^n(z+1)^n}{1 - z(z+1)} \\ &= \frac{(1-2z)(1-pz-pz^2) + (1-p)z^3(z^2+z)^n}{1 - z - z^2}. \end{aligned}$$

Thus $P(V_n) = \log 2 - O((3/4)^n)$.

Remark 3.3. In the Example 3.2, the term $1 - pz - pz^2$ shows that the unstable τ and shift invariant set $Y = \{y \in S^{\mathbf{Z}}; y_i y_{i+1} \neq 00\}$ exists and $(Y, \tau) \simeq (Y, \sigma)$.

References

[1] W. Parry and M. Pollicott: Zeta functions and the periodic orbit structure of hyperbolic dynamics. Astérisque, no. 187–188, Société mathématique de France, France (1990).

[2] S. Wolfram: Theory and Applications of Cellular Automata. World Scientific, Singapore (1984).