On semicontinuous solutions for general Hamilton-Jacobi equations

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1. Introduction. This is an announcement of our recent work [16], where detailed proofs are given as well as extensions. We do not give the proof of statements. We consider the initial value problem for the Hamilton-Jacobi equation of form

(1a) $u_t + H(x, u_x) = 0 \quad \text{in } (0,T)\times\mathbf{R}^n,$

(1b) $u(0,x) = u_0(x), \qquad x \in \mathbf{R}^n,$

where $u_t = \partial u/\partial t$ and $u_x = (\partial_{x_1}u, \ldots, \partial_{x_n}u)$, $\partial_{x_i}u = \partial u/\partial x_i$; $\infty \geq T > 0$ is a fixed number. Our main goal is to find a suitable notion of solution when u_0 is discontinuous. The theory of viscosity solutions initiated by Crandall and Lions [6] yields the global solvability of the initial value problem by extending the notion of solutions when u_0 is continuous (cf. [8, Chap.10], [15], [2]). In fact, if initial data u_0 is bounded, uniformly continuous, it is well-known [6], [15] that the initial value problem (1a)-(1b) admits a unique global (uniformly) continuous viscosity solution when H is enough regular, for example H satisfies the Lipschitz conditions

(2a) $|H(x,p) - H(x,q)| \leq C|p-q|$

(2b) $|H(x,p) - H(y,p)| \leq C(1+|p|)|x-y|.$

We only refer to [2], [15] and [7] for the basic theory of viscosity solutions. The notion of viscosity solution has been extended to semicontinuous functions. This is very important to prove the existence of solutions without appealing hard estimates. Such a method is first introduced by [13]. However, if u_0 is, for example, upper semicontinuous, a classical semicontinuous viscosity solution may not be unique.

Recently to overcome this inconvenience, Barron and Jensen [3] introduced another notion of viscosity solutions for semicontinuous functions when the Hamiltonian $H = H(x,p)$ is concave in p and proved the existence and the uniqueness of their solution for (1a), (1b) for bounded (from above), upper semicontinuous initial data u_0. Their solution is now called a bilateral viscosity solution [1]. For later development of the theory as well as other approaches we refer to [1] and references cited there. However, their theory is limited for concave H. (In [3] H is assumed to be convex but they consider the terminal value problem which is easily transformed to the initial value problem with concave Hamiltonian by setting $T-t$ by t.)

In this paper we introduce a new notion of a solution which is unique for a given initial upper semicontinuous initial data. For (1a), (1b) we consider auxiliary problem

(3a) $\psi_t - \psi_y H(x, -\psi_x/\psi_y) = 0 \quad \text{in } (0,T)\times\mathbf{R}^{n+1},$

(3b) $\psi(0,x,y) = \psi_0(x,y), \qquad (x,y) \in \mathbf{R}^n\times\mathbf{R}.$

The equation (3a) is called the level set equation for the evolution of the graph of u of (1a). In fact, if a level set of a solution ψ of (3a) is given as the graph of a function $v = v(t,x)$, then v must solve (1a). For given upper semicontinuous initial data $u_0 : \mathbf{R}^n \to \mathbf{R}\cup\{-\infty,+\infty\}$, shortly $u_0 \in USC(\mathbf{R}^n)$, we take

(4) $\psi_0(x,y) = -\min\{\mathrm{dist}((x,y),K_0),\ 1\},$

where

(5) $K_0 = \{(x,y)\in\mathbf{R}^n\times\mathbf{R};\ y \leq u_0(x)\}.$

We solve (3a), (3b) and set

(6) $\overline{u}(t,x) = \sup\{y\in\mathbf{R};\ \psi(t,x,y)\geq 0\},$

where ψ is the continuous viscosity solution of (3a), (3b). We call $\overline{u}$ an *L-solution* of (1a), (1b). Such a solution uniquely exists globally in time under suitable condition on H.

Theorem 1. *Assume that the recession function*

(7) $H_\infty(x,p) = \lim_{\lambda\downarrow 0}\lambda H(x,p/\lambda), \quad x\in\mathbf{R}^n,\ p\in\mathbf{R}^n$

exists and that H satisfies (2a), (2b). *Then there exists a global unique L-solution for an arbitrary*

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$u_0 \in USC(\mathbf{R}^n)$.

One may relax the assumptions on H (cf. Remark 11 and [16]) but in this paper we shall always assume (2a), (2b) and (7). These assumptions guarantee that the singularity at $\psi_y = 0$ in (3a) is removable if we restrict ψ satisfying $\psi_y \le 0$. Moreover, (3a), (3b) admits a unique global solution for any bounded, uniformly continuous initial data $\psi_0 = \psi_0(x,y)$ which is nonincreasing in y. (The monotonicity of the solution ψ in y is preserved for $t > 0$.)

2. Comparison and uniqueness. Since a solution of (3a), (3b) enjoys a comparison principle, so does an L-solution (1a), (1b).

Theorem 2 (comparison). *Let u and v be the L-solution of* (1a), (1b) *with initial data u_0 and v_0, respectively, where $u_0, v_0 \in USC(\mathbf{R}^n)$. If $u_0 \le v_0$ on $\mathbf{R}^n$, then $u \le v$ on $(0,T) \times \mathbf{R}^n$.*

In the definition of an L-solution the specific form of ψ_0 given by (4) is not important.

Theorem 3 (uniqueness). *Assume that ψ_0 is a bounded uniformly continuous function such that $\{\psi_0 \ge 0\} = K_0$ and that $y \mapsto \psi_0(x,y)$ is nonincreasing. Let ψ be the solution of* (3a), (3b). *Then*

$$\tilde{u}(t,x) = \sup\{y \in \mathbf{R};\ \psi(t,x,y) \ge 0\},\qquad t \in (0,T),\ x \in \mathbf{R}^n$$

agrees with the L-solution of (1a), (1b).

The key observation for the proof is that the set $\{\psi \ge 0\}$ $(= \{(t,x,y);\ \psi(t,x,y) \ge 0\}$ depends only on K_0 and is independent of the choice of ψ_0. This is a typical uniqueness property of a level set equation. It is based on invariance of solution under the change of the dependent variable as stated below (which is slightly more general than stated in references [9], [10], [4], [11], [14] since θ need not be continuous).

Lemma 4 (invariance). *Assume that ψ is a subsolution* (*resp. supersolution*) *of* (3a). *Assume that θ is upper* (*resp. lower*) *semicontinuous and nondecreasing. Assume that $\theta \not\equiv -\infty$* (*resp. $\theta \not\equiv +\infty$*). *Then the composite function $\theta \circ \psi$ is also a subsolution* (*resp. supersolution of* (3a)).

If $\{\psi \ge 0\}$ were a bounded set, a comparison principle for (3a), (3b) and Lemma 4 would yield the uniqueness of $\{\psi \ge 0\}$ as in [10], [4], [11]. However, since $\{\psi \ge 0\}$ is unbounded, we actually argue as in [14] to get the uniqueness of $\{\psi \ge 0\}$.

3. Consistency. We shall compare other notion of solutions.

Theorem 5. *Let $\overline{u}$ be the L-solution of* (1a), (1b) *with $u_0 \in USC(\mathbf{R}^n)$. Then $\overline{u}$ be a viscosity solution of* (1a) *provided that $\overline{u}$ does not take $\pm\infty$.*

Sketch of the proof. Let ψ be the solution of (3a), (3b) with ψ_0 in (4). By Lemma 4 the function $I^- \circ \psi$ is a subsolution of (3a), where $I^-(\sigma) = 0$ for $\sigma \ge 0$ and $I^-(\sigma) = -\infty$ for $\sigma < 0$. From this it is easy to see that $\overline{u}$ is a viscosity subsolution.

To prove that $\overline{u}$ is a viscosity supersolution we need to use the fact that $y \mapsto \psi(x,y)$ is nonincreasing. This implies that the lower semicontinuous envelope $(\overline{u})_*$ of $\overline{u}$ equals

$$\underline{u}(t,x) = \inf\{y \in \mathbf{R};\ (t,x,y) \in \overline{\{\psi < 0\}}\}\qquad t \in (0,T),\ x \in \mathbf{R}^n.$$

Since $I^+ \circ (\psi + 1/m)$ is a supersolution of (3a) by Lemma 4, we see, by stability as $m \to \infty$, that

$$\Psi(t,x,y) = \begin{cases} \infty & \text{for } (t,x,y) \in \operatorname{int}\{\psi \ge 0\}, \\ 0 & \text{for } (t,x,y) \in \overline{\{\psi < 0\}} \end{cases}$$

is a subsolution of (3a), where $I^+(\sigma) = 0$ for $\sigma \le 0$ and $I^+(\sigma) = \infty$ for $\sigma > 0$. Thus $\underline{u}$ is a supersolution. □

Theorem 6. *Assume that u_0 is bounded, uniformly continuous. Then the bounded, uniformly continuous viscosity solution u of* (1a), (1b) *is an L-solution.*

This follows from Theorem 3 by choosing

$$\psi = ((y - u(t,x)) \wedge M) \vee M \quad \text{for} \quad M = \sup|u|.$$

Theorem 7. *Assume that $p \mapsto H(x,p)$ is concave. Let $\overline{u}$ be the L-solution of* (3a), (3b) *with $u_0 \in USC(\mathbf{R}^n)$ and $\sup u_0 < \infty$. Then $\overline{u}$ is a bilateral viscosity solution with initial data u_0.*

For the proof we use the property that the bilateral viscosity solution is given as a monotone limit of continuous viscosity solution [3]. Thus the proof is reduced to the next lemma.

Lemma 8. *Assume that $u_{0\varepsilon} \downarrow u_0 \in USC(\mathbf{R}^n)$ with $u_{0\varepsilon}$ which is Lipschitz in $\mathbf{R}^n$. Assume that $u_{0\varepsilon} \ge u_{0\varepsilon'} + \varepsilon - \varepsilon'$ for $\varepsilon > \varepsilon' > 0$. Let u_ε be the solution of* (1a), (1b) *with $u_0 = u_{0\varepsilon}$. Then $\lim_{\varepsilon\to 0} u_\varepsilon$ is an L-solution of* (1a), (1b) (*so that it agrees with $\overline{u}$*).

The sequence $u_{0\varepsilon}$ is easily constructed by setting $u_{0\varepsilon} = u_0^\varepsilon + \varepsilon$ with sup-convolution u_0^ε of u_0.

4. Right accessibility. It is not clear in what sense the initial value is attained for L-solutions (unless initial data is continuous.) Since the viscosity solution of (3a), (3b) with ψ_0 in (4) is continuous up to $t = 0$, the set $\{\psi \geq 0\}$ is closed in $[0, T) \times \mathbf{R}^n \times \mathbf{R}$ so that

$$u_0(x) \geq \varlimsup_{\substack{t \downarrow 0 \\ y \to x}} \overline{u}(t, y). \tag{8}$$

However, in general it is not clear whether there is a sequence $t_m \to 0$, $y_m \to x$ such that

$$u_0(x) = \lim_{m \to \infty} \overline{u}(t_m, y_m). \tag{9}$$

We call this last property the right accessibility as in [5]. Since $\overline{u}$ is upper semicontinuous in $[0, T) \times \mathbf{R}^n$, the property (9) is equivalent to $u_0(x) = (\overline{u}|_{(0,T) \times \mathbf{R}^n})_*\ (0, x)$. We give a simple criterion for right accessibility.

Lemma 9. *Assume that $F \in C(\mathbf{R}^N)$ is positively homogeneous of degree one. Let A be a closed convex set in $\mathbf{R}^N$. Let w be the L-solution of*

$$w_t + F(w_z) = 0, \quad z \in \mathbf{R}^N,\ t > 0; \quad w|_{t=0} = w_0.$$

with $w_0(z) = 0$, $z \in A$ and $\sup\{w_0(z); \operatorname{dist}(z, A) \geq \delta\} < 0$ for $\delta > 0$. Then

$$w(t, z) = \begin{cases} 0 & z \in A + tW_\alpha \\ < 0 & \text{otherwise.} \end{cases}$$

Here

$$W_\alpha = \{z \in \mathbf{R}^N;\ \sup_{|p|=1} (z \cdot p - \alpha(p)) \leq 0\},$$
$$\alpha(p) = -F(-p).$$

The set W_α is often called the Wulff shape with respect to α if α is positive. The set W_α may be empty. For example if $F(p) = |p|$, then $W_\alpha = \emptyset$. If we consider (1a), (1b) with $H(p) = |p|$ and $u_0(x) = 0$, $x = 0$; $u_0(x) = -\infty$, $x \neq 0$, then the L-solution $u(t, x) = -\infty$ for all $t > 0$. This is easy to prove since $v(t, x) = -mt - m|x|$ is a continuous solution for all $m > 0$ and $u \leq v$ by Theorem 2. For this problem (9) for u is not fulfilled.

Theorem 10. *If H is positively homogeneous degree of one, and independent of x, then an L-solution is right accessible for any $u_0 \in USC(\mathbf{R}^n)$ if and only if $W_\alpha \neq \emptyset$.*

Remark 11. Our results up to §3 can be generalized for more general equation

$$u_t + H(x, u, u_x) = 0,$$

when H fulfills

(i) $H \in C(\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n)$ and H_∞ exists;

(ii) There exists a modulus m_1 that satisfies
$$|qH(x, y - p/q) - qH(x', y', -p/q)| \leq m_1((|x - x'| + |y - y'|)(|p| + |q| + 1))$$
for all $x, y, x', y' \in \mathbf{R}^n$, $p \in \mathbf{R}^n$, $q < 0$;

(iii) For each $C_1 > 0$ there exists a modulus m_2 such that
$$|qH(x, y - p/q) - q'H(x, y, -p'/q')| \leq m_2(|p - p'| + |q - q'|)$$
for all $x \in \mathbf{R}^n$, $y \in \mathbf{R}$, $p, p' \in \mathbf{R}^n$, $q, q' < 0$ satisfying $|p|, |p'|, |q|, |q'| \leq C_1$;

(iv) $y \mapsto H(x, y, p)$ is nondecreasing.

A typical example of H satisfying these assumptions is $a(x)\sqrt{b + |p|^\beta}$ and a is Lipschitz and $0 \leq \beta \leq 1$, $b \geq 0$.

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