"Hasse principle" for $GL_2(D)$

By Takashi Ono

Department of Mathematics, The Johns Hopkins University, Baltimore, MD 21218, U.S.A.

(Communicated by Shokichi Iyanaga, M. J. A., Sept. 13, 1999)

1. Statement of a theorem. Let D be a Euclidean domain and $G = GL_2(D)$, the group of invertible 2×2 matrices over D.[0] We shall prove that

(1.1) **Theorem.** *$Ш(G) = 1$, i.e., G enjoys the "Hasse principle".*[1]

(1.2) **Remark.** Thanks to an excellent idea of M. Mazur, to prove (1.1) it is enough to verify that

$$\text{(1.3)} \qquad \mathrm{End}_c(G) = \mathrm{Inn}(G),$$

where the left hand side is the set of all endomorphisms of G preserving conjugacy classes of G.[1] Thus for each $F \in \mathrm{End}_c(G)$, and $A \in G$, we have

$$\text{(1.4)} \qquad F(A) \sim A, \quad \text{i.e.,} \quad F(A) = PAP^{-1}, \quad P \text{ depending on } A.$$

Given an $F \in \mathrm{End}_c(G)$ we connect two elements A, B of G by *a* string according to the rule:

$$\text{(1.5)} \qquad A \text{—} B \Longleftrightarrow \exists P \in G \quad \text{so that} \quad F(A) = PAP^{-1} \text{ and } F(B) = PBP^{-1}.$$

Note that A—B is not, a priori, an equivalence relation defined on G.[2] Even so, this relation is very useful to prove the Hasse principle $Ш(G) = 1$. Note also that the relation (1.5) depends only on F modulo $\mathrm{Inn}(G)$.

2. Generators for G. Before proving (1.1), let us gather some basic facts on $G = GL_2(D)$, D being a Euclidean domain. Denote by D^* the group of invertible elements of D. Let N, M_λ ($\lambda \in D$, $\lambda \neq 0$), D_μ ($\mu \in D^*$, $\mu \neq 1$) be elements of G defined by

$$\text{(2.1)} \quad N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad D_\mu = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}.$$

It is well-known and easy to prove that

$$\text{(2.2)} \qquad G \text{ is generated by } N,\ M_\lambda,\ D_\mu: \quad G = \langle N, M_\lambda, D_\mu \rangle,$$

We will use repeatedly the following equalities on $P = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in G$.

$$\text{(2.3)} \ PNP^{-1} = (\det P)^{-1} \begin{pmatrix} yt - xz & x^2 - y^2 \\ t^2 - z^2 & xz - yt \end{pmatrix},$$

$$\text{(2.4)} \quad M_\lambda PNP^{-1} = (\det P)^{-1} \times \begin{pmatrix} yt - xz + \lambda(t^2 - z^2) & x^2 - y^2 + \lambda(xz - yt) \\ t^2 - z^2 & xz - yt \end{pmatrix},$$

$$\text{(2.5)} \quad D_\mu PNP^{-1} = (\det P)^{-1} \begin{pmatrix} yt - xz & x^2 - y^2 \\ \mu(t^2 - z^2) & \mu(xz - yt) \end{pmatrix}.$$

3. Proof of the theorem.

Step (I). To prove that N—M_λ. Since we can adjust a given F in $\mathrm{End}_c(G)$ by elements of $\mathrm{Inn}(G)$, We may assume that

$$\text{(3.1)} \qquad \begin{cases} F(M_\lambda) = M_\lambda, \\ F(N) = PNP^{-1}, \quad P \in G. \end{cases}$$

Our problem is to find $P_0 \in G$ so that

$$\text{(3.2)} \qquad \begin{cases} F(M_\lambda) = M_\lambda = P_0 M_\lambda {P_0}^{-1}, \\ F(N) = PNP^{-1} = P_0 N {P_0}^{-1}. \end{cases}$$

Put

$$\text{(3.3)} \qquad P = \begin{pmatrix} x & y \\ z & t \end{pmatrix}, \quad P_0 = \begin{pmatrix} 1 & y_0 \\ 0 & 1 \end{pmatrix}.$$

Clearly P_0, with any $y_0 \in D$, meets the first equality of (3.2). As for the second equality of (3.2), in view of (2.3) for P and P_0 we are forced to set $y_0 = (yt - xz)/(xt - yz)$ and then we should verify the equality (3.2) which boils down to a single equality:

$$\text{(3.4)} \qquad \det(P) = xt - yz = t^2 - z^2$$

as a little calculation shows. To get (3.4), we must use seriously the assumption that F is a homomor-

[0] Needless to say, D may be any commutative field.

[1] As for unexplained notation and facts in this paper, see [1].

[2] This reminds me somehow a children's string game CAT'S CRADLE, or AYATORI in Japanese. One can play this game on any group G once an endomorphism $F \in \mathrm{End}_c(G)$ is chosen.

phism: $F(M_\lambda N) = F(M_\lambda)F(N)$. In other words, we have

$$Q(M_\lambda N)Q^{-1} = M_\lambda PNP^{-1}, \quad \text{with some } Q \in G. \tag{3.5}$$

Take the trace of both sides of (3.5) and use (2.4). Then (3.4) follows miraculously.

Step (II). To Prove that $N — D_\mu$. As in Step (I), we may assume that

$$\begin{cases} F(D_\mu) = D_\mu, \\ F(N) = PNP^{-1}, \quad P \in G. \end{cases} \tag{3.6}$$

Again we must find $P_0 \in G$ so that

$$\begin{cases} F(D_\mu) = D_\mu = P_0 D_\mu {P_0}^{-1} \\ F(N) = PNP^{-1} = P_0 N {P_0}^{-1}. \end{cases} \tag{3.7}$$

Put

$$P = \begin{pmatrix} x & y \\ z & t \end{pmatrix}, \quad P_0 = \begin{pmatrix} x_0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.8}$$

Clearly P_0, with any $x_0 \in D^*$, meets the first equality of (3.7). As for the second equality of (3.7), in view of (2.3) for p and p_0 we are forced to set $x_0 = (x^2 - y^2)/(xt - yz)$ and then we should verify the equality (3.7) which boils down to a single equality:

$$xz - yt = 0 \tag{3.9}$$

as a little calculation shows. To get (3.9), we must use again the property $F(D_\mu N) = F(D_\mu)F(N)$. We have then

$$Q(D_\mu N)Q^{-1} = D_\mu PNP^{-1}, \quad Q \in G. \tag{3.10}$$

Take the trace of both sides of (3.10) and use (2.5). Then (3.9) follows again.

Step (III). Combining (I), (II), we found, for any $F \in \mathrm{End}_c(G)$, $\lambda(\neq 0) \in D$, $\mu(\neq 1) \in D^*$, matrices P, Q in G so that

$$\begin{cases} F(N) = PNP^{-1} = QNQ^{-1}, \\ F(M_\lambda) = PM_\lambda P^{-1}, \\ F(D_\mu) = QD_\mu Q^{-1}. \end{cases} \tag{3.11}$$

Here it is important to note that the matrix Q above does not depend on μ up to scalars. In fact, let $X = {Q_\mu}^{-1} Q_{\mu'}$, with $F(N) = Q_\mu N {Q_\mu}^{-1} = Q_{\mu'} N {Q_{\mu'}}^{-1}$, $F(D_\mu) = Q_\mu D_\mu {Q_\mu}^{-1}$, $F(D_{\mu'}) = Q_{\mu'} D_{\mu'} {Q_{\mu'}}^{-1}$. Then

$$X = \begin{pmatrix} x & y \\ y & x \end{pmatrix}, \; x^2 - y^2 \in D^*.$$

Then by comparing the traces of both sides of $F(D_{\mu\mu'}) = F(D_\mu)F(D_{\mu'})$, one verifies that $X = xI$, and so one can assume that $Q_\mu = Q_{\mu'} = Q$.

From the first line of (3.11), we infer that

$$R = Q^{-1}P = \begin{pmatrix} a & b \\ b & a \end{pmatrix}. \tag{3.12}$$

Adjusting F modulo $\mathrm{Inn}(G)$, the last two lines of (3.11) imply that

$$\begin{cases} F(M_\lambda) = RM_\lambda R^{-1}, \\ F(D_\mu) = D_\mu. \end{cases} \tag{3.13}$$

Taking the traces of both sides of $F(D_\mu M_\lambda) = F(D_\mu)F(M_\lambda)$, we get, after a little calculation using $\lambda \neq 0$, $\mu \neq 1$ and $a^2 - b^2 \neq 0$,

$$ab = 0. \tag{3.14}$$

In other words,

$$R = aI \quad \text{or } bN, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.15}$$

So from (3.11), (3,13), (3.15), we may assume that either

$$\begin{cases} F(N) = N = {}^tN, \\ F(M_\lambda) = NM_\lambda N^{-1} = {}^tM_\lambda, a = 0, \\ F(D_\mu) = D_\mu = {}^tD_\mu. \end{cases} \tag{3.16}$$

or

$$\begin{cases} F(N) = N, \\ F(M_\lambda) = M_\lambda, \quad b = 0, \\ F(D_\mu) = D_\mu. \end{cases} \tag{3.17}$$

If (3.16) was the case, we would have $NM_\lambda D_\mu \sim F(NM_\lambda D_\mu) = F(N)F(M_\lambda)F(D_\mu) = N\,{}^tM_\lambda D_\mu = {}^t(D_\mu M_\lambda N)$ and, on taking the traces, we get $\lambda\mu = \lambda$, or $\lambda(\mu - 1) = 0$, contradicting our assumption on λ, μ. So (3.17) shows that our original F is an inner automorphism, i.e., the Hasse principle $\text{Ш}(G) = 1$ holds. □

References

[1] T. Ono: "Shafarevich-Tate sets" for profinite groups. Proc. Japan Acad., **75A**, 97–98 (1999).

[2] H. Wada: "Hasse principle" for $SL_n(D)$. Proc. Japan Acad., **75A**, 67–69 (1999).