Some formulas in $\mathfrak{sl}_3$ weight systems

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Abstract: We exhibit some simplest nontrivial relations among the values of the $\mathfrak{sl}_3$ weight systems which seem to generate a large portion of the kernel of the weight system.

1. Background and notation. A chord diagram (CD, for short) on the circle is a graph with univalent and trivalent vertices such that all the univalent vertices are placed along a fixed circle (called the Wilson loop of the diagram) and all the trivalent vertices are given cyclic orderings of the adjacent three edges. Two chord diagrams are identified up to diffeomorphisms of the union of the uni-trivalent graph and the Wilson loop which preserve the orientation of the loop and the cyclic orderings of the trivalent vertices. We say a CD is connected when every component of the uni-trivalent graph has at least one univalent vertex on the Wilson loop. When uni-trivalent graphs are pictured below, we always assume that the anticlockwise ordering is chosen at each trivalent vertex.

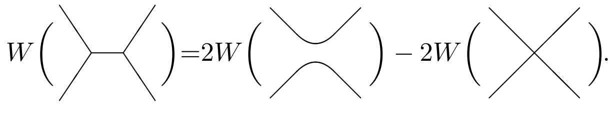
Let $\mathcal{A}$ be the vector space (over the complex numbers) generated by all connected CDs modulo STU relations (illustrated in the Appendix). The antisymmetry AS and the IHX relations (illustrated in the Appendix) also hold in $\mathcal{A}$; they correspond to the antisymmetry and the Jacobi identity of the Lie algebra respectively. This space $\mathcal{A}$ is introduced to combinatrially describe the space of Vassiliev's knot invariants of finite type: each finite type invariant corresponds to a linear functional on $\mathcal{A}$ [1][4]. It becomes a commutative and cocommutative Hopf algebra in a natural way and has certain universal property with respect to the universal enveloping algebras of semisimple Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra and b an ad-invariant nondegenerate bilinear form b. Kontsevich constructed a natural map $W : \mathcal{A} \to Z(\mathcal{U}(\mathfrak{g}))$ where $Z(\mathcal{U}(\mathfrak{g}))$ is the center of the universal enveloping algebra. He defined W by multiplying the Lie algebra elements corresponding to the univalnet vertices along the Wilson loop and summing up all such configurations of elements. In this procedure elements are contracted by the given bilinear form along edges and by the Lie algebra structure constants around trivalent vertices [3]. This map is called the weight system associated with the Lie algebra and the bilinear form.

The same contraction as above can be applied to any uni-trivalent graphs Γ with univalent veritces placed on some fixed points. The contraction produces an element in the tensor product $\mathfrak{g}^{\otimes \ell}$, where ℓ is the number of the fixed points. We denote this element still by the same symbol $W(\Gamma)$. If we decompose a connected CD D into a collection of such uni-trivalent graphs $\Gamma_1, \Gamma_2, \dots \Gamma_k$ by cutting some edges, then we can obtain $W(D)$ by contracting $W(\Gamma_1)W(\Gamma_2)\dots W(\Gamma_k)$ along those edges using the given bilinear form.

How far the weight system as a map $W : \mathcal{A} \to Z(\mathcal{U}(\mathfrak{g}))$ is from being an isomorphic depends on each Lie algebra, and is generally unknown (See problem 5.3 in [1]). In their paper [2] S. V. Chmutov and A. N. Varchenko studied this map when the Lie algebra is $\mathfrak{sl}_2$ and obtained a recurrent formula for the conbinatorial computation of this map. Their main basic formula is the following.

Theorem 0 (Theorem 6 of Chmutov-Varchenko[2]). *Let W be the weight system associaed to $\mathfrak{sl}_2$ and the matrix trace as the invariant bilinear form. Then for any three uni-trivalnet graphs differing only in the pictured part as below, the following relation holds*:



2. Theorems. If we try to do the same thing as described above for the Lie algebra $\mathfrak{sl}_3$, the same basic formula fails to hold; these three terms become linearly independent. Simplest nontrivial formulas

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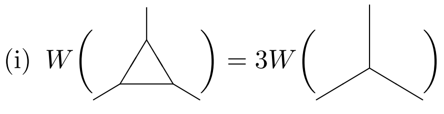
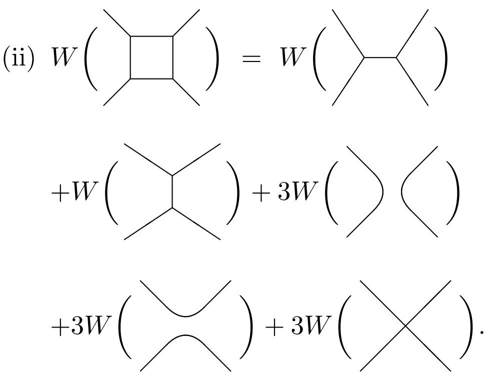
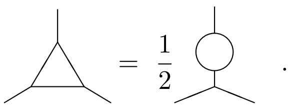
in $\mathfrak{sl}_3$ weight systems are the following.

Theorem 1. *Let W be the weight system associated to $\mathfrak{sl}_3$ and the matrix trace as the invariant bilinear form. Then for uni-trivalent graphs differing only in the pictured part as below, the following relations hold*:

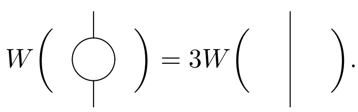




Remark. The AS and IHX relation imply

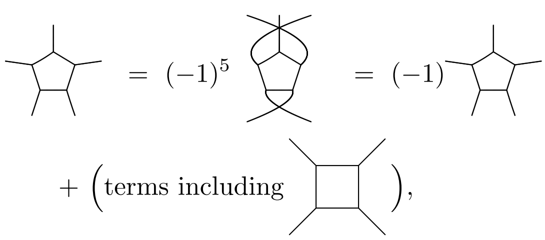


Hence Theorem 1 (i) is an immediate corollary to



For Theorem 1 (ii), the authors do not know a simple proof (See the indication of proof below).

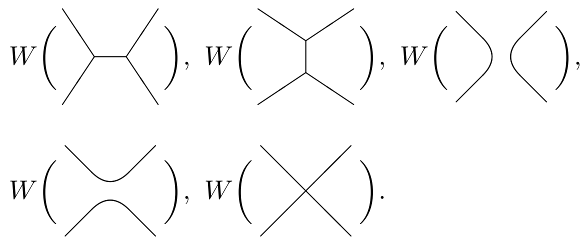
Similar relation for the five-legged-wheel may be obtained using Theorem 1 (ii), since:



In the above formula, we take the mirror image using the AS relation five times to check the first equality, and we change the order of the five legs using the IHX relation to check the second.

These formulas (and the remark above) express the W value of graphs with loops by the values of graphs witout loops. These are the simplest in the sense that the value of these graphs without loops are independent if the number of the univalent vertices is less than five.

Theorem 2. *Let W be as in* Theorem 1. *The values of the following graphs are linearly independent in $\mathfrak{sl}_3^{\otimes 4}$.*



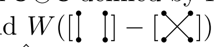
3. Indication of Proofs. The relations in Theorem 1 are first found through calculation by hand. Once they are found, it is straightfoward in principle to verify them. In fact, since both-hand-sides of (ii) are elements of $\mathfrak{sl}_3^{\otimes 4}$, the proof is obtained by calculating the corresponding tensors and comparing them. Similarly, the proof of Theorem 2 is obtained by showing that corresponding system of linear equations have no solutions. The only problem is that their dimensions are large. We used Mathematica to confirm the results. Any other relevant computer programs should work as well.

More efficient proof is to look at the decomposition of the vector space $\mathfrak{sl}_3^{\otimes n}$ into irreducible components via the action of the symmetry group $\mathfrak{S}_n$ and the adjoint action of $\mathfrak{sl}_3$.

Let $\mathcal{A}$(4pts) be the vector space spanned by uni-trivalent graphs with 4 univalent vertices placed on fixed four points, subject to the AS and IHX relations. The weight system $W : \mathcal{A}(\text{4pts}) \to (\mathfrak{sl}_3^{\otimes 4})^{\mathfrak{sl}_3}$ intertwines the obvious actions of $\mathfrak{S}_4$.

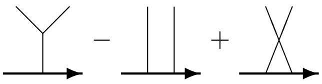
In $\mathcal{A}$(4pts), as a representation space of $\mathfrak{S}_4$, the element [ ]+[ ]+[ ] generate a trivial representation $\mathbf{1}$ of $\mathfrak{S}_4$, and the elements [ ] − [ ], [ ] − [ ] generate a two-dimensional irreducible representation, say $\mathbf{2}$, and the elements [ ], [ ] generate another isomorphic two-dimensional representation, say $\mathbf{2}'$. To see Thoerem 2 is true, it suffices to see that the W images of these five elements are linearly independent in $(\mathfrak{sl}_3^{\otimes 4})^{\mathfrak{sl}_3}$.

Denoting the adjoint representation of $\mathfrak{sl}_3$ by $\mathbf{8}$ and identifying the dual $\mathbf{8}^*$ with $\mathbf{8}$, we may regard $(\mathfrak{sl}_3^{\otimes 4})^{\mathfrak{sl}_3} = \mathrm{Hom}_{\mathfrak{sl}_3}(\mathbf{8} \otimes \mathbf{8}, \mathbf{8} \otimes \mathbf{8})$. With respect to the decomposion $\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus (\mathbf{8} \times 2) \oplus \mathbf{10} \oplus \hat{\mathbf{10}} \oplus \mathbf{27}$, $W([\ ]) \in \mathrm{Hom}_{\mathfrak{sl}_3}(\mathbf{8} \otimes \mathbf{8}, \mathbf{8} \otimes \mathbf{8})$ is the projector to

$\mathbf{0} \oplus \mathbf{8} \subset \mathbf{8} \times 2$, which is the (-1)-eigenspace of the permutation P on $\mathbf{8} \otimes \mathbf{8}$ defined by $P(x \otimes y) = y \otimes x$. On the other hand  is the projector to $(\mathbf{0} \oplus \mathbf{8}) \oplus \mathbf{10} \oplus \hat{\mathbf{10}}$. It follows that $\mathbf{2}$ and $\mathbf{2}'$ go to different subspaces of $(\mathfrak{sl}_3^{\otimes 4})^{\mathfrak{sl}_3}$ as representations of $\mathfrak{S}_4$, showing the linear independence in Theorem 2.

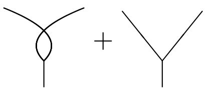
Appendix. The STU relation:

The STU relation is generated by the following linear combinations of three diagrams differing only in the common part of the diagrams as illustrated below:



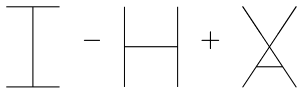
AS relarion:

The AS relation is generated by the sums of two diagrams differing only in the common part of the diagrams in the manner illustrated below:



The IHX relation:

The IHX relation is generated by the following linear combinations of three diagrams differing only in the common part of the diagrams as illustrated below:



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References

[1] D. Bar-Natan: On the Vassiliev knot invariants. Topology, **34**, 423–472 (1995).

[2] S. V. Chmutov snf A. N. Varchenko: Remarks on the Vassiliev knot invariants comming from sl_2. Topology, **36**, 153–178 (1997).

[3] M. Kontsevich: Vassiliev's knot invariants. Adv. Soviet Math., **16**, Part 2, 137–150 (1993).

[4] V. A. Vassiliev: Cohomology of knot spaces. Theory of Singularities and Its Applications (ed. V. I. Arnold). Adv. Soviet Math., **1**, 23–69 (1990).