The Bergman kernel on weakly pseudoconvex tube domains in $\mathbf{C}^2$

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1. Introduction. In this note, we announce a result in [14] of an asymptotic expansion of the Bergman kernel for certain class of weakly pseudoconvex tube domains of finite type in $\mathbf{C}^2$. We also give an analogous result of the Szegö kernel for the same class of tube domains.

Let Ω be a domain with smooth boundary in $\mathbf{C}^n$. The Bergman space $B(\Omega)$ is the subspace of $L^2(\Omega)$ consisting of holomorphic L^2-functions on Ω. The Bergman projection is the orthogonal projection $\mathbf{B} : L^2(\Omega) \to B(\Omega)$. It is known that the projection $\mathbf{B}$ can be represented by using some integral kernel:

$$\mathbf{B}f(z) = \int_\Omega K(z,w)f(w)dV(w) \quad \text{for } f \in L^2(\Omega).$$

where $K : \Omega \times \Omega \to \mathbf{C}$ is the *Bergman kernel* of the domain Ω and dV is the Lebesgue measure on Ω. In this paper we restrict the Bergman kernel on the diagonal of the domain and study the boundary behavior of $K(z) = K(z,z)$.

There are many studies about the boundary behavior of the Bergman kernel. First we consider the Bergman kernel $K(z)$ of a bounded strictly pseudoconvex domain Ω. Hörmander [12] showed that the limit of $K(z)d(z-z^0)^{n+1}$ at $z^0 \in \partial\Omega$ equals the determinant of the Levi form at z^0 times $n!/4\pi^n$, where d is the Euclidean distance. Diederich [5], [6] obtained analogous results for the first and mixed second derivatives of $K(z)$. Moreover C. Fefferman [8] and Boutet de Monvel and Sjöstrand [3] gave a very strong result of the boundary behavior of Bergman kernel. The Bergman kernel of Ω can be expressed in the following:

$$(1.1) \qquad K(z) = \frac{\varphi(z)}{r(z)^{n+1}} + \psi(z)\log r(z),$$

where $r \in C^\infty(\overline{\Omega})$ is a defining function of Ω (i.e. $\Omega = \{r > 0\}$ and $|dr| > 0$ on $\partial\Omega$) and φ, $\psi \in C^\infty(\overline{\Omega})$ can be expanded asymptotically with respect to r.

On the other hand, there are not so strong results in the weakly pseudoconvex case. Let us recall important studies in this case. Many estimates of the size of the Bergman kernel have been obtained (see the references in [1]). In particular Catlin [4] gave a complete estimate from above and below for domains of finite type in $\mathbf{C}^2$. Recently Boas, Straube and Yu [1] and Diederich and Herbort [7] computed a boundary limit in the sense of Hörmander for a large class of domains of finite type in $\mathbf{C}^n$ on a nontangential cone. Though the above studies about *estimate* and *boundary limit* are detailed, clear asymptotic formula like (1.1) is yet to be obtained. In this note we give an asymptotic expansion of the Bergman kernel for certain class of weakly pseudoconvex tube domains of finite type in $\mathbf{C}^2$. Gebelt [9] and Haslinger [11] recently computed certain asymptotic formulas for the special cases, but the method of our expansion is different from theirs.

Our main idea used to analyse the Bergman kernel is to introduce certain real blowing-up. Since the set of strictly pseudoconvex points are dense on the boundary of the domain of finite type, it is a serious problem to resolve the difficulty caused by the influence of strictly pseudoconvex points near z^0. This problem can be avoided by restricting the argument on a non-tangential cone in the domain. We surmount the difficulty in the case of certain class of tube domains by blowing up at the weakly pseudoconvex point z^0 and introducing two new variables. The Bergman kernel can be developed asymptotically in terms of these variables.

Our method of the computation is based on the studies [8], [3], [2], [20]. Our starting point is certain integral representation of the Bergman kernel in [17],

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[19], [21], [10]. We consider $\{(z_1, z_2) \in \mathbf{C}^2; \mathrm{Im} z_2 > g[\mathrm{Im} z_1]^{2m}\}$ with $g > 0$, $m = 2, 3, \cdots$ as a *model* domain. After introducing the blowing-up to this representation, we compute the asymptotic expansion by using the stationary phase method. For the above computation, it is necessary to localize the Bergman kernel near a weakly pseudoconvex point. This localization can be obtained in a fashion similar to the case of some class of Reinhardt domains ([2], [20]).

2. Statement of main result. Given a function $f \in C^\infty(\mathbf{R})$ satisfying that

$$(2.1)\qquad \begin{cases} f'' \geq 0 \text{ on } \mathbf{R} \text{ and } f \text{ takes the form in some neighborhood of 0:} \\ f(x) = x^{2m} g(x) \text{ where } m = 2, 3, \cdots, \\ \qquad g(0) > 0 \text{ and } x g'(x) \leq 0. \end{cases}$$

Let $\omega_f \subset \mathbf{R}^2$ be a domain defined by $\omega_f = \{(x, y); y > f(x)\}$. Let $\Omega_f \subset \mathbf{C}^2$ be the tube domain over ω_f, i.e.,

$$\Omega_f = \mathbf{R}^2 + i\omega_f.$$

Let $\pi : \mathbf{C}^2 \to \mathbf{R}^2$ be the projection defined by $\pi(z_1, z_2) = (\mathrm{Im} z_1, \mathrm{Im} z_2)$. Set $O = (0, 0)$. It is easy to check that Ω_f is a pseudoconvex domain; moreover $z^0 \in \partial\Omega_f$, with $\pi(z^0) = O$, is a weakly pseudoconvex point of type $2m$ (or $2m - 1$) in the sense of Kohn or D'Angelo and $\partial\Omega_f \backslash \pi^{-1}(O)$ is strictly pseudoconvex near z^0.

Now we introduce the transformation σ, which plays a key role on our analysis. Set $\Delta = \{(\tau, \varrho); 0 < r \leq 1, \varrho > 0\}$. The transformation $\sigma : \overline{\omega_f} \to \overline{\Delta}$ is defined by

$$(2.2)\qquad \sigma : \begin{cases} \tau = \chi(1 - f(x)/y), \\ \varrho = y, \end{cases}$$

where the function $\chi \in C^\infty([0, 1))$ satisfies the conditions: $\chi'(u) \geq 1/2$ on $[0, 1]$, and $\chi(u) = u$ for $0 \leq u \leq 1/3$ and $\chi(u) = 1 - (1 - u)^{\frac{1}{2m}}$ for $1 - 1/3^{2m} \leq u \leq 1$. Then $\sigma \circ \pi$ is the transformation from $\overline{\Omega}$ to $\overline{\Delta}$.

The transformation σ induces an isomorphism of $\omega_f \cap \{x \geq 0\}$ (or $\omega_f \cap \{x \leq 0\}$) on to Δ. The boundary of ω_f is transfered by σ in the following: $\sigma((\partial\omega_f) \backslash \{O\}) = \{(0, \varrho); \varrho > 0\}$ and $\sigma^{-1}(\{(\tau, 0); 0 \leq \tau \leq 1\}) = \{O\}$. This indicates that σ is the real blowing-up of $\partial\omega_f$ at O, so we may say that $\sigma \circ \pi$ is the blowing-up at the weakly pseudoconvex point z^0. Moreover σ patches the coordinates (τ, ϱ) on ω_f, which can be considered as the polar coordinates around O. We call τ the angular variable and ϱ the radial variable, respectively. Note that if z approaches some strictly (resp. weakly) pseudoconvex points, $\tau(\pi(z))$ (resp. $\varrho(\pi(z))$) tends to 0 on the coordinates (τ, ϱ).

The following theorem asserts that the singularity of the Bergman kernel of Ω_f at z^0, with $\pi(z^0) = O$, can be essentially expressed in terms of the polar coordinates (τ, ϱ).

Theorem 2.1. *The Bergman kernel of Ω_f takes the form in some neighborhood of z^0 :*

$$(2.3)\qquad K(z) = \frac{\Phi(\tau, \varrho^{\frac{1}{m}})}{\varrho^{2+\frac{1}{m}}} + \tilde{\Phi}(\tau, \varrho^{\frac{1}{m}}) \log \varrho,$$

where $\Phi \in C^\infty((0, 1] \times [0, \varepsilon))$ and $\tilde{\Phi} \in C^\infty([0, 1] \times [0, \varepsilon))$ with some $\varepsilon > 0$.

Moreover Φ is written in the form on the set $\{\tau > \alpha\varrho^{\frac{1}{2m}}\}$ with some $\alpha > 0$: for every nonnegative integer μ_0

$$(2.4)\quad \Phi(\tau, \varrho^{\frac{1}{m}}) = \sum_{\mu=0}^{\mu_0} c_\mu(\tau)\varrho^{\frac{\mu}{m}} + R_{\mu_0}(\tau, \varrho^{\frac{1}{m}})\varrho^{\frac{\mu_0}{m} + \frac{1}{2m}},$$

where

$$(2.5)\qquad c_\mu(\tau) = \frac{\varphi_\mu(\tau)}{\tau^{3+2\mu}} + \psi_\mu(\tau) \log \tau,$$

for φ_μ, $\psi_\mu \in C^\infty([0, 1])$, φ_0 is positive on $[0, 1]$ and R_{μ_0} satisfies $|R_{\mu_0}(\tau, \varrho^{\frac{1}{m}})| \leq C_{\mu_0}[\tau - \alpha\varrho^{\frac{1}{2m}}]^{-4-2\mu_0}$ for some positive constant C_{μ_0}.

Let us describe the asymptotic expansion of the Bergman kernel K in more detail. Considering the meaning of the variables τ, ϱ, we may say that each expansion with respect to τ or $\varrho^{\frac{1}{m}}$ is induced by the strict or weak pseudoconvexity, respectively. Actually the expansion (2.5) has the same form as that of Fefferman (1.1). By (2.4), (2.5), in order to see the characteristic influence of the weak pseudoconvexity on the singularity of the Bergman kernel K, it is sufficient to argue about K on an *approach region* $\mathcal{U}_\alpha = \{z \in \mathbf{C}^2; \tau \circ \pi(z) > \alpha^{-1}\}$ $(\alpha > 1)$. This is because $\mathcal{U}_\alpha$ is the *widest* region where the coefficients $c_\mu(\tau)'$s are bounded. The region $\mathcal{U}_\alpha$ seems deeply connected with the admissible approach regions appeared in [18]. We remark that on the region $\mathcal{U}_\alpha$, the exchange of the expansion variable $\varrho^{\frac{1}{m}}$ for $r^{\frac{1}{m}}$, where r is a defining function of Ω_f (e.g. $r(x, y) = y - f(x)$), gives no influence on the form of the expansion on the region $\mathcal{U}_\alpha$.

Now let us compare the asymptotic expansion (2.3) on $\mathcal{U}_\alpha$ with Fefferman's expansion (1.1). The essential difference between them only appears in the expansion variable (i.e. $r^{\frac{1}{m}}$ in (2.3) and r in (1.1)). A similar phenomenon occurs in subelliptic estimates for the $\overline{\partial}$-Neumann problem. As is well-known, the finite-type condition is equivalent to the condition that a subelliptic estimate holds, i.e.,

$$|||\phi|||_\epsilon^2 \le C(\|\overline{\partial}\phi\|^2 + \|\overline{\partial}^*\phi\|^2 + \|\phi\|^2) \quad (\epsilon > 0)$$

(refer to [16] for the details). Let ϵ_0 be the best possible order of subellipticity. In two dimensional case, $\epsilon_0 = \frac{1}{2}$ in the strictly pseudoconvex case and $\epsilon_0 = \frac{1}{2m}$ in the weakly pseudoconvex case of type $2m$. The difference between these two cases only appears in the value of ϵ_0. From this viewpoint, our expansion (2.3) seems to be a natural generalization of Fefferman's expansion (1.1) in the strictly pseudoconvex case.

Remark 1. From the viewpoint of the studies [1], [7], let us consider the limit of $\varrho^{2+\frac{1}{m}}K(z)$ at z^0. This limit on a nontangential cone is

$$\begin{aligned} &c_0(1) \\ &= \varphi_0(1) \\ &= \frac{g(0)^{\frac{1}{m}}}{(4\pi)^2}\Gamma(2+\frac{1}{m}) \cdot \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-w^{2m}+vw} dw \right]^{-1} dv. \end{aligned}$$

The above integral seems impossible to be changed into simpler form (see [15]). We remark that if we do not restrict the region for the approach, the above limit is $c_0(\tau)$, which depends on the angular variable τ.

Remark 2. The idea of the blowing-up σ is originally introduced in the study of the Bergman kernel of the domain $\{z \in \mathbf{C}^n; \sum_{j=1}^n |z_j|^{2m_j} < 1\}$ ($m_j \in \mathbf{N}$, $m_n \ne 1$) in [13]. Since the above domain has high homogeneity, the asymptotic expansion with respect to the radial variable does not appear.

Remark 3. If we consider the Bergman kernel on the region $\mathcal{U}_\alpha$, then we can remove the condition $xg'(x) \le 0$ in (2.1). Namely even if the condition $xg'(x) \le 0$ is not satisfied, we can still obtain (2.3), (2.4) in the theorem where c_μ's are bounded on $\mathcal{U}_\alpha$. But, for our method of computation, the condition $xg'(x) \le 0$ is necessary to obtain the asymptotic expansion with respect to τ.

3. In the case of the Szegö kernel.

Let Ω_f be a tube domain satisfying the condition in Section 2. Let $H^2(\Omega_f)$ be the subspace of $L^2(\Omega_f)$ consisting of holomorphic functions F on Ω_f such that

$$\sup_{\epsilon>0} \int_{\partial\Omega_f} |F(z_1, z_2 + i\epsilon)|^2 d\sigma(z) < \infty,$$

where $d\sigma$ is the measure on $\partial\Omega_f$ given by the Lebesgue measure on $\mathbf{C} \times \mathbf{R}$ when we identify $\partial\Omega_f$ with $\mathbf{C} \times \mathbf{R}$ (by the map $(z, t + if(\mathrm{Im} z)) \mapsto (z,t)$). The Szegö projection is the orthogonal projection $\mathbf{S} : L^2(\partial\Omega_f) \to H^2(\Omega_f)$ and we can write

$$\mathbf{S}F(z) = \int_{\partial\Omega_f} S(z,w)F(w)d\sigma(w),$$

where $S : \Omega_f \times \Omega_f \to \mathbf{C}$ is the *Szegö kernel* of the domain Ω_f. We are interested in the restriction of the Szegö kernel on the diagonal, so we write $S(z) = S(z,z)$.

We also give an asymptotic expansion of the Szegö kernel of Ω_f. The theorem below can be obtained in a fashion similar to the case of the Bergman kernel.

Theorem 3.1. *The Szegö kernel of Ω_f takes the form in some neighborhood of z^0 :*

$$S(z) = \frac{\Phi^S(\tau, \varrho^{\frac{1}{m}})}{\varrho^{1+\frac{1}{m}}} + \tilde{\Phi}^S(\tau, \varrho^{\frac{1}{m}}) \log \varrho,$$

where $\Phi^S \in C^\infty((0,1] \times [0,\varepsilon))$ and $\tilde{\Phi}^S \in C^\infty([0,1] \times [0,\varepsilon))$, with some $\varepsilon > 0$.

Moreover Φ^S is written in the form on the set $\{\tau > \alpha\varrho^{\frac{1}{2m}}\}$ with some $\alpha > 0$: for every nonnegative integer μ_0

$$\Phi^S(\tau, \varrho^{\frac{1}{m}}) = \sum_{\mu=0}^{\mu_0} c_\mu^S(\tau)\varrho^{\frac{\mu}{m}} + R_{\mu_0}^S(\tau, \varrho^{\frac{1}{m}})\varrho^{\frac{\mu_0}{m}+\frac{1}{2m}},$$

where

$$c_\mu^S(\tau) = \frac{\varphi_\mu^S(\tau)}{\tau^{2+2\mu}} + \psi_\mu^S(\tau) \log \tau,$$

for φ_μ^S, $\psi_\mu^S \in C^\infty([0,1])$, φ_0^S is positive on $[0,1]$ and $R_{\mu_0}^S$ satisfies $|R_{\mu_0}^S(\tau, \varrho^{\frac{1}{m}})| \le C_{\mu_0}^S[\tau - \alpha\varrho^{\frac{1}{2m}}]^{-3-2\mu_0}$ for some positive constant $C_{\mu_0}^S$.

Note added in proof. Recently the full paper was published in [22].

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