3-manifold groups and property T of Kazhdan

By Koji FUJIWARA

Mathematical Institute, Tohoku University, Aoba, Aramaki, Aoba-ku, Sendai, 980-8578

(Communicated by Heisuke HIRONAKA, M. J. A., Sept. 13, 1999)

Abstract: Suppose that M is a compact, orientable three-manifold such that each piece of the canonical decomposition along embedded spheres, discs and tori admits one of the eight geometric structures of three-manifolds in the sense of Thurston. Let G be a subgroup of $\pi_1(M)$. If G has property T in the sense of Kazhdan, then G is finite.

Key words: Property T of Kazhdan; three-manifold groups; property FA of Serre.

1. Introduction. A discrete group G has *property* FA if when it acts on a simplicial tree by isometries the action has a (global) fixed point in the tree, [6]. Kazhdan defined that G has *property* T if the trivial representation of G is an isolated point in the set of equivalent classes of irreducible unitary representations, equipped with the Fell topology (see [2]). It is known that property T implies property FA [1], [4]. An important class of discrete groups with property T is given by lattices in connected, simple Lie groups of rank greater than 1, for example; $SL_n(\mathbf{Z}), n \geq 3$.

Let M be a compact, orientable 3-manifold. Suppose that no connected components of ∂M are homeomorphic to S^2. Note that this assumption is not very restrictive to us since if there exists a boundary component which is homeomorphic to S^2, then we can remove it by gluing a 3-ball to M without changing the fundamental group.

It is known that there is a canonical way to decompose M into pieces by cutting M along embedded spheres, discs and tori, which is called *the canonical decomposition* of M (see [3],[5]). If each of the resulting pieces admits one of the eight geometric structures, then we say that M is *geometric* [7].

Theorem 1. *Let M be a compact, orientable three-manifold. Suppose that no connected components of the boundary of M are homeomorphic to S^2. Assume that M is geometric. Let G be a subgroup in $\pi_1(M)$. If G has property T, then G is finite.*

2. Proof of the Theorem 1.

Lemma 2. *Let M be a compact, orientable three-manifold. Suppose that no connected components of the boundary of M are homeomorphic to S^2. Let G be a subgroup in $\pi_1(M)$. Suppose G has property FA. Then*

(1) *there is a piece N in the decomposition of M such that a conjugate of G is a subgroup of $\pi_1(N)$.*

(2) *If we additionally assume that M is geometric, then the piece N satisfies one of the following.*

(i) *N is spherical, and G is finite.*

(ii) *N is hyperbolic.*

Proof. (1). Let $M = \cup_i M_i$ be the canonical decomposition of M. Then $\pi_1(M)$ admits a decomposition into $\pi_1(M_i)$'s in terms of free product of amalgamation and/or HNN-extension. Since G has property FA, there must be a piece N in M_i such that a conjugate of G is in $\pi_1(N)$.

(2). For the simplicity of notation, we denote the conjugate of G obtained in (1) by G as well. Since N admits one of the eight geometric structures, one of the following holds.

(i) The geometry of N is spherical.

(i) N is a Seifert manifold.

(iii) The geometry of N is hyperbolic.

(iv) The geometry of N is Euclidean.

(v) N is a surface bundle over a circle.

We discuss each case. Note that if G is trivial, then we have nothing to prove.

(i) This is one of our conclusion. Note that $\pi_1(N)$ is finite.

(ii) There is an exact sequence

$$0 \to \mathbf{Z} \to \pi_1(N) \to \pi_1^{orb}(\Sigma) \to 1,$$

where Σ is the underlying 2-orbifold. Let H be the image of G in the exact sequence. Then H has property FA as well. But a subgroup in $\pi_1^{orb}(\Sigma)$ has

1991 Mathematics Subject Classification. Primary 57M05; Secondary 22D10, 20E08.

property FA only when it is in $\mathbf{Z}/n\mathbf{Z}$ which corresponds to one of the singular points of the underlying 2-orbifold. If G is non-trivial, then it follows that G is isomorphic to $\mathbf{Z}$, which does not satisfy property FA. Therefore G is trivial.
(iii) This is one of our conclusion.
(iv) In this case $\pi_1(N)$ is amenable. But amenable groups have property T only when they are finite. So G is finite. But in this case G has no torsion, so G is trivial.
(v) There is an exact sequence

$$1 \to \pi_1(\Sigma) \to \pi_1(N) \to \mathbf{Z} \to 0,$$

where Σ is the fiber. Let H be the image of G in $\mathbf{Z}$. Since G has property FA, so does H, therefore H must be trivial. This implies that G is in fact in $\pi_1(\Sigma)$. But subgroups in a surface group have property FA only when they are trivial. Thus G is trivial. □

We now prove the Theorem 1.

Proof . Since we assume that G has property T, it also has property FA, [1], [4]. We apply the Lemma 2 to G. To finish the proof, we need to exclude the case that N is hyperbolic in the conclusion of the Lemma 2. To argue by contradiction we suppose N is hyperbolic and G is not trivial. Then G is a discrete subgroup in $SO(3,1)$. It is known that a countable subgroup in $SO(3,1)$ has property T only when it is precompact, [2], [8]. Therefore G is finite, and in fact trivial since G has no torsion. This is a contradiction. □

Acknowledgements. This work is completed during the author was at a Rigidity Theory metting in Paris in June 1998 partially supported by C.N.R.S. He would like to thank Professor A.Valette for useful informations.

References

[1] R. Alperin: Locally compact groups acting on trees and property T. Monatsh. Math., **93**, no.4, 261–265 (1982).
[2] P. de la Harpe and A. Valette: La proprit (T) de Kazhdan pour les groupes localement compacts. Astérisque, no. 175 pp. 1–158 (1989).
[3] W. H. Jaco and P. Shalen: Seifert fibered spaces in 3-manifolds. Mem. Amer. Math. Soc., **21**, no.220, pp. 1–192 (1979).
[4] Y. Watatani: Property T of Kazhdan implies property FA of Serre. Math. Japon., **27** no.1, 97–103 (1982).
[5] P. Scott: The geometries of 3-manifolds. Bull. London Math. Soc., **15**, 401–487 (1983).
[6] J. P. Serre: Trees. Springer, Berlin, pp. 1–142 (1980).
[7] W. Thurston: Three-dimensional geometry and topology. vol.1. Princeton Math. Ser., 35. Princeton University Press, Princeton, pp. 1–311 (1997).
[8] R. J. Zimmer: Kazhdan groups acting on compact manifolds. Invent. Math., **75** no.3, 425–436 (1984).