Dynamics of composite functions

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Abstract: Let f and g be two transcendental entire functions. In this paper, mainly by using Iversen's theorem on the singularities, we studied the dynamics of composite functions. We have proved that the Fatou sets of $f \circ g$ and $g \circ f$ have the same dynamical properties.

Key words: Entire function; complex dynamics; composite functions.

1. Introduction. Let $f(z)$ be a nonlinear entire function. The sequence of the iterates of f is denoted by

$$f^{n+1} = f^n \circ f$$

where $f^0 = id$, $f^1 = f$. We define $F = F(f)$ to be the largest open set in which the iterates of f form a normal family, and

$$J = J(f) = C - F(f).$$

They are called the Fatou set and Julia set of f, respectively.

Suppose U is a component of the Fatou set of f, U is called a wandering domain if $f^m(U) \cap f^n(U) = \phi$ for $m \neq n$. If U is not wandering, we call U a pre-periodic component of f. That is, $f^n(f^m(U)) = f^m(U)$ for $n, m \geq 0$. If $m = 0$, we call U a periodic component of f. D. Sullivan, see, e.g. [9] proved that the Fatou set of any rational function has no wandering domain; I. N. Baker and others, see, e.g. [2] gave examples to show that transcendental entire functions may have wandering domains. In [1], it is known that functions which have only a finite number of asymptotic and critical values have no wandering domain. I. N. Baker and A. P. Singh [3] in 1995 proved that if $p(z)$ is a non-constant entire function and $g(z) = a + be^{2\pi iz/c}$, where a, b and c are non-zero constants, such that $g \circ p$ has no wandering domain, then so does $p \circ g$. We have generalized this and proved that if f and g are two given transcendental entire functions, then $f \circ g$ has wandering domains if and only if $g \circ f$ does. Moreover, we have shown that the dynamics of $f \circ g$ and $g \circ f$ are very similar.

2. The lemmas and main results.

Lemma 2.1. (Iversen's theorem, see [7]) *Let F be a Riemann surface of parabolic type over the w plane, and let $w = w_0$ be an arbitrary point in the plane. Further assume that $\delta > 0$ and that w_1 is an interior point of the surface F with $|w_1 - w_0| = \delta$. Then it is possible to find a continuous curve L that joins the points w_1 and w_0 without leaving the disk $|w - w_0| < \delta$ and that with the possible exception of the end point w_0 consists of nothing but interior points of the surface F.*

Concerning the components of the Fatou set, we have the following two lemmas:

Lemma 2.2 (I. N. Baker [1]). *Let f be a transcendental entire function. Then every unbounded component U of $F(f)$ is simply connected.*

Lemma 2.3 (I. N. Baker [1]). *Let f be a transcendental entire function. Then any pre-periodic Fatou component U is simply connected, and therefore any multiple-connected Fatou component is bounded and wandering.*

Theorem 2.1. *Suppose that f and g are entire functions. Then $g \circ f$ has no wandering domain, if and only if $f \circ g$ has no wandering domain.*

Theorem 2.2. *Suppose that f and g both are transcendental entire functions. Then $f \circ g$ contains a Schröder domain if and only if $g \circ f$ does. In addition, similar conclusions hold for a Leau domain, Siegel disc, Baker domain and Böttcher domain.*

3. Proofs of Theorems.

3.1. Proof of Theorem 2.1.

Proof. We first assume that $g \circ f$ has no wandering domain. Let $K = f \circ g$ and $H = g \circ f$. Then we have $H \circ g = g \circ K$. Suppose on the contrary that K has a sequence of wandering do-

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mains $\{U_i\}$, where $K(U_i) \subset U_{i+1}$ and $U_m \cap U_n = \phi$ for $m \neq n$. Then $g(U_j)$ and $g(U_k)$ are pairwisely disjoint for $j \neq k$; otherwise we have $K(U_j) \cap K(U_k) = f(g(U_j)) \cap f(g(U_k)) \neq \phi$, which contradicts the fact that $\{U_i\}$ is a sequence of wandering domains of K. Let $V_j = g(U_j)$. Then it follows that $\{V_j\}$ are pairwisely disjoint. Now from

$$(H \circ g)(U_j) = (g \circ K)(U_j),$$

we have $H(V_j) \subset g(U_{j+1})$ and hence $H(V_j) \subset V_{j+1}$. By Montel's theorem, $\{H^n\}$ is normal in each V_k since $\{H^n\}$ takes no values which lie in V_k for $n > k+1$. So, V_k belongs to the Fatou set of H. We finally want to show that each V_k is a component of $F(H)$ which leads to a contradiction to our hypothesis.

Case 1. If $\beta \in \partial V_k$ and β has the form $\beta = g(\alpha)$, where $\alpha \in \partial U_k$, then $\beta \in J(H)$. Since U_k is a component of $F(K)$, thus $\partial U_k \subset J(K)$. Therefore, α is a limit point of the repelling periodic points z_n of K, say $K^{v_n}(z_n) = z_n$. Since $H^{v_n} \circ g = g \circ K^{v_n}$ for all n, one obtains $H^{v_n}(g(z_n)) = g(z_n)$. Hence $g(z_n)$ is a periodic point of H. Moreover, it is easily to check that $g(z_n)$ is a repelling periodic point of H and $\{g(z_n)\}$ tends to $g(\alpha) = \beta$ as n tends to infinity, so $\beta \in J(H)$.

Case 2. Suppose that $\beta \in \partial V_k$ but β does not belong to $J(H)$. Then actually, we have a component $W_k \subset F(H)$ such that $V_k \subset W_k$. We want to show that $V_k = W_k$, except for at most one point. From our assumption, β is not a limit point of $J(H)$, hence it is not a limit point of $g(\partial U_k)$. Thus, there exists a disc $D = D(\beta, r)$ with $r > 0$, which contains no points of $g(\partial U_k \setminus \{\infty\})$. Since $\beta \in \partial V_k$, we can choose $w' \in D(\beta, r)$ such that $w' = g(z')$ for $z' \in U_k$. By Iversen's theorem, we can find a path joining w' and β in D except perhaps for a point β in D so that the inverse branch of g is continuous on the path which lies in U_k and it never hits ∂U_k. This implies that $\beta \in g(U_k)$ which is a contradiction. For the exceptional case, there exists an asymptotic path Γ in U_k such that along Γ, g has an asymptotic value β. U_k is now unbounded, since it contains an asymptotic path. By Lemma 2.2, U_k is simply-connected. In addition, by Lemma 2.3, W_k is simply connected, because according to the hypothesis, W_k is not a wandering domain. Hence there exists conformal maps ϕ and ψ from the unit disc Δ onto U_k and W_k respectively. Define $h = \psi^{-1} \circ g \circ \phi$ so that $h(\Delta) \subset \Delta$. Clearly, it is sufficient to prove that $\Delta \setminus h(\Delta)$ contains at most one point. By a result of Beurling (see e.g. [5]), there exists a set $A \subset [0, 2\pi]$ of capacity zero with the property that if $\theta \notin A$, then there exists $a_\theta \in \partial U_k \setminus \{\infty\}$ such that $\phi(re^{i\theta}) \to a_\theta$ as $r \to 1$. It follows that $g(\phi(re^{i\theta})) \to g(a_\theta) \in \partial W_k \setminus \{\infty\}$, which belongs to the Julia set of H, and hence $|h(re^{i\theta})| \to 1$ as $r \to 1$, provided $\theta \notin A$. A result of Lohwater (see e.g. [5]) now implies that $\Delta \setminus h(\Delta)$ contains at most one point. Hence in this case, $V_k = W_k$ except for at most one point.

Thus, we have shown that V_k is a wandering component of H. This is a contradiction and hence the theorem is proved.

3.2. Proof of Theorem 2.2.

Proof. Let $K = f \circ g$ and $H = g \circ f$. Suppose U is a periodic component of $F(K)$. Without loss of generality, we may assume that U is forward invariant, otherwise we can consider its iterates K^n. According to [4], $U \setminus K(U)$ contains at most one point, so we can further assume that $K(U)$ is a component of $F(K)$ and $K(U) = U$ for simplicity. It is easy to see that

(1) $$g \circ K = H \circ g,$$

and hence $g(U) = (H \circ g)(U)$. Let $V = g(U)$, we have $H(V) = V$ and V does not contain any repelling periodic point of H. By Montel's theorem, H is normal in $g(U)$. Similar to Theorem 2.1, one can show that V is a forward invariant component of $F(H)$. Now, we have several cases to consider:

Case 1. Assume that U is a Leau domain of K. Then for all $z \in U$, $K^n(z) \to z_0 \in \partial U$, where z_0 is an indifferent fixed point of K. Since $g \circ K = H \circ g$, hence it is obvious that $g(z_0)$ is a fixed point of H. From (1), we have

$$g'(z_0)K'(z_0) = H'(g(z_0))g'(z_0).$$

Since $K'(z_0) = (f \circ g)'(z_0) = f'(g(z_0))g'(z_0) \neq 0$, this implies $g'(z_0) \neq 0$ and hence $|K'(z_0)| = |H'(g(z_0))| = 1$. Therefore, $g(z_0)$ is an indifferent fixed point of H. Since $z_0 \in J(K)$, there exists a sequence of repelling periodic points $\{\alpha_n\}$ of K of period u_n such that $\alpha_n \to z_0$, and $g(\alpha_n) = H^{v_n}(g(\alpha_n))$. It is obvious that $g(\alpha_n)$ are repelling periodic points of H, and $g(\alpha_n) \to g(z_0)$, hence $g(z_0) \in J(H)$ and so $g(z_0) \in \partial V$. Thus V is a Leau domain of H.

The proofs that U is a Schröder domain or a Böttcher domain are similar to the above case.

Case 2. Assume that U is a Siegel disc of K. We suppose that $K^n(z) \to \phi(z)$ in U, where ϕ is analytic and non-constant in U. Then $H^n(z) \to (g \circ \phi)(z)$ for all $z \in V$ and V is a forward invariant component of H. Hence H contains a Siegel disc.

Case 3. Assume that U is a Baker domain. From the fact that $g(U) = V$ and $f(V) = U$, we can conclude that V has to be a Baker domain since there exists a one-one relationship between U and V.

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