Corestriction principle in non abelian Galois cohomology

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Introduction. Let k be a field of characteristic 0, G a linear algebraic group defined over k. We are interested only in linear algebraic k-groups, so the adjective "linear" will be omitted. It is well-known (see e.g. [16]) that if G is commutative, then for any finite extension k' of k, there is the so-called corestriction map $Cores_{G,k'/k}$ (which will be denoted also by $Cores_G$ to emphasize the group G, when the fields k', k are fixed):

$$Cores_G : \mathrm{H}^q(k', G) \to \mathrm{H}^q(k, G),\ q \geq 0,$$

where $\mathrm{H}^q(L, H)$ denotes the Galois cohomology $\mathrm{H}^q(Gal(\bar{L}/L), H(\bar{L}))$ for a L-group H defined over a field L of characteristic 0 (or a perfect field L). However if G is not commutative, there is no such a map in general and, as far as we know, the most general sufficient conditions are given in [14], under which such a map can be constructed. The Corestriction Theory constructed there has many applications to theory of algebras, representation theory and related questions. In this paper we are interested in the following natural question about the corestriction map.

Assume that there is a map, which is functorial in k:

$$\alpha : \mathrm{H}^p(k, G) \to \mathrm{H}^q(k, T),$$

where T is a commutative k-group, G a non-commutative k-group, i.e., α gives rise to a morphism of functors $(k \mapsto \mathrm{H}^p(k, G)) \to (k \mapsto \mathrm{H}^q(k, T))$ (cf. also [17], Section 6.1). By restriction, for any finite extension k'/k we have a functorial map

$$\alpha' : \mathrm{H}^p(k', G) \to \mathrm{H}^q(k', T).$$

Question. *When does* $Cores_T$ (Im (α')) $\subset$ Im (α)?

If the answer is affirmative for all k', we say that *the Corestriction Principle holds* for (the image of) the map α. One defines similar notion for the kernel of a map $\beta : \mathrm{H}^p(k, T) \to \mathrm{H}^q(k, G)$.

We say that the map $\alpha : \mathrm{H}^p(k, G) \to \mathrm{H}^q(k, T)$ is *standard* if it is obtained as a connecting map from the exact cohomology sequence associated with an exact sequence of k-groups involving G and T. For example, let

$$1 \to A \to B \to C \to 1,$$

be an exact sequence of k-groups, where A is considered as a normal k-subgroup of B. Then

$$\mathrm{H}^i(k, A) \to \mathrm{H}^i(k, B),\ i = 0, 1,$$

and

$$\mathrm{H}^0(k, C) \to \mathrm{H}^1(k, A)$$

are standard maps. In general, C is just a quotient space and may not be a group. If A is a central subgroup of G, then C is a group, and one may define a connecting standard map $\mathrm{H}^1(k, C) \to \mathrm{H}^2(k, A)$.

It is worth mentioning that in some particular cases, the above question has an affirmative answer unconditionally and the *Norm Principle* is said to hold if it holds for $p = q = 0$ (which approves the adjective *norm*). There are some examples to support this principle, for example, by considering reduced norm in division algebras, the Scharlau norm principle ([18, 20]), etc. A new kind of Corestriction Principle over local and global fields has been found by P. Deligne [5], Prop. 2.4.8, which, in the case of characteristic 0 and in notations of abelian Galois cohomology [1], [12], Appendix B), says that the Corestriction Principle for images holds for the map

$$ab^0_G : \mathrm{H}^0(k, G) \to \mathrm{H}^0_{ab}(k, G).$$

However, given any natural numbers $n \geq 2$, $r \geq 1$, Rosset and Tate have constructed in [15] an example of a field E containing the group μ_n of n-th roots of 1, a finite Galois extension F of E of degree r, and an element x of $K_2(F)$, which is a symbol, such that the image of x via the trace

$$Tr_{F/E} : K_2F \to K_2E$$

is a sum of at least r symbols. From this they derive a symbol algebra of degree n over F, considered as an element of $\mathrm{H}^2(F, \mu_n)$, such that its

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image via the corestriction

$$Cores_{F/E}: \mathrm{H}^2(F, \mu_n) \to \mathrm{H}^2(E, \mu_n)$$

is *not* a symbol. Therefore the question above has a negative answer for the standard map

$$\Delta : \mathrm{H}^1(E, \mathrm{PGL}_n) \to \mathrm{H}^2(E, \mu_n).$$

Despite of this, we will see that in many interesting cases, the Corestriction Principle for standard maps hold, especially in the case the field of definition is a local or global field of characteristico . In particular, our first main result in Section 1 (Theorem 1.6) can be considered as a generalization of the statement: *The norm of a symbol is a norm.* In certain sense, it is a cohomological complement to the well-known result by Lenstra [10] and Tate [19] that for a local or global field F, every element of $K_2(F)$ is a symbol, and it extends the result of deligne (above) to higher dimensions. If the base field is an arbitrary field of characteristic 0, we discuss the relation between the corestriction principles for various types of standard maps.

1. Corestriction Principle in non-abelian cohomology: local and global fields. We use the notion and results from the Borovoi-Kottwitz theory of abelian Galois cohomology of algebraic groups as presented in [1]-[3](see also [12], Appendix B, for a survey). We recall briefly that for a connected reductive group G defined over a field k of characteristic 0 with a maximal k-torus T, let $\tilde{G}$ be the simply connected covering of the semisimple part of G with maximal torus $\tilde{T}$, which is projected into a subtorus of T via the isogeny $\tilde{G} \to G' := [G, G]$. One can define a complex of tori $T^\bullet = (\tilde{T} \to T)$ where T (resp. $\tilde{T}$) is in degree 0 (resp.-1). Then

$$\mathrm{H}^i_{ab}(k, G) := \mathscr{H}^i(k, T^\bullet)$$

where $\mathscr{H}^i$ denotes the Galois hypercohomology of the complex $T^\bullet$. Then it was shown that $\mathrm{H}^i_{ab}(k, G)$, $i \geq 0$, satisfy usual functorial properties of a cohomology theory, and there exist functorial homomorphism and map, respectively

$$ab^0_G : \mathrm{H}^0(k, G) \to \mathrm{H}^0_{ab}(k, G),$$

$$ab^1_G : \mathrm{H}^1(k, G) \to \mathrm{H}^1_{ab}(k, G).$$

Our first main result of this section is the following

1.1. Theorem. *Let k be a local or global field of characteristic 0, G a connected k-group, T a connected commutative k-group and $\alpha : \mathrm{H}^p(k, G) \to \mathrm{H}^q(k, T)$ a standard map. Assume that G is a central extension of T if $p = q = 2$ where the 2-cohomology is defined as in* [7]. *Then for $0 \leq p \leq q \leq 2$ the Corestriction Principle holds for the image of α.*

The proof uses main results in the Galois cohomology of algebraic groups over local and global fields, due to Kneser and Harder [9], [8] and also abelianized Galois cohomology due to Borovoi [1]-[3].

1.2. Remarks. It follows from the construction of ab^2_G of [2], p. 228, that this map satisfies the Corestriction Principle for images for any field k of characteristic 0 and any connected reductive k-group G.

2) Another proof of Theorem 1.1 (without using abelianized Galois cohomology) follows from main results of Section 2.

To be complete, together with the Corestriction Principle for the *images* of standard maps, we need also to consider the validity of this principle for *kernels* of standard maps. Namely for a standard map

$$\alpha : \mathrm{H}^p(k, T) \to \mathrm{H}^q(k, G),$$

where T, G are connected k-groups with T commutative, and for a finite extension k' of k with the corestriction map $Cores_T : \mathrm{H}^p(k', T) \to \mathrm{H}^p(k, T)$, we ask

Question. *When does* $Cores_T(\mathrm{Ker}\,(\alpha \otimes k')) \subset \mathrm{Ker}\,(\alpha)$?

By using Theorem 1.1 it is easy to see that in the case k is a global or a global field of characteristic 0, one is reduced to considering the case $p = q = 1$. We have the following affirmative result for local and global fields of characteristic 0, and it is our second main result in this section.

1.3. Theorem. *Let k be a local or global field of characteristic* 0 *and T a connected commutative k-subgroup of a connected k-group G. Then the Corestriction Principle holds for the kernel of the standard map $\alpha : \mathrm{H}^1(k, T) \to \mathrm{H}^1(k, G)$.*

The proof uses z-extensions, which are the same as *cross-diagram* due to Ono [13]. Recall that a connected reductive k-group H is a z-extension of a k-group G if H is an extension of G by an *induced* k-torus Z, such that the derived subgroup (semisimple part) $[H, H]$ of H is simply connected. From the proof of Theorem 1.3 we can deduce the following

1.4. Corollary. *The Corestriction Principle*

for kernels of the standard maps $\mathrm{H}^1(k, T) \to \mathrm{H}^1(k, G)$, *where* T *and* G *are connected groups over a field* k *of characteristic* 0, T *is commutative, holds if and only if the same holds for all pairs* (T, G) *with* T *a maximal torus of a simply connected almost simple* k*-group* G, *all defined over* k.
We derive the following consequence, which is a slight generalization of a result of Deligne [5], Proposition 2.4.8.

1.5. Theorem. *With the above notation, assume that* G *is a connected reductive* k*-group. For any finite extension* k' *of a local or global field* k *there is a canonical norm map*

$$G(k')/\pi(G_0(k')) \to G(k)/\pi(G_0(k)).$$

From Theorem 1.1 and 1.3 we derive the following main result of this section.

1.6. Theorem. (Corestriction Principle) *Let* G, T *be connected linear algebraic groups, where* T *is commutative, all defined over local or global field* k *of characteristic* 0. *Assume that* α_k : $\mathrm{H}^p(k, G) \to \mathrm{H}^q(k, T)$ (*resp.* $\beta_k : \mathrm{H}^q(k, T) \to \mathrm{H}^p(k, G)$) *is a standard map. Then for any finite extension* k'/k *we have*

$$Cores_{k'/k}(\mathrm{Im}\,(\alpha_{k'})) \subset \mathrm{Im}\,(\alpha_k),$$

$$\left(resp.\ Cores_{k'/k}(\mathrm{Ker}\,(\beta_{k'})) \subset \mathrm{Ker}\,(\beta_k\right).$$

2. Corestriction Principle in non-abelian cohomology : arbitrary field of characteristic 0. In this section we will discuss some relation between the validity of Corestriction Principles for standard maps of various type. As applications we apply the results obtained to give new proof of a result of Deligne that we used in Section 1.

Let k be a field of characteristic 0 and α : $\mathrm{H}^p(k, G) \to \mathrm{H}^q(k, T)$ be a standard map, where $p = 0, 1$, $q \leq p + 1$, G and T are connected reductive k-groups, T is a torus. Denote by $\tilde{G}$ (resp. $\bar{G}$) the simply connected covering (resp. the adjoint) group of the semisimple part of G, $\tilde{F} = \mathrm{Ker}\,(\tilde{G} \to \bar{G})$, $F' = \mathrm{Ker}\,(G' \to \bar{G})$, where G' is the semisimple part of G. We consider the following statements.

a) *The Corestriction Principle for images holds for any such* α.

b) *The Corestriction Principle for images holds for* $\mathrm{H}^p(k, \bar{G}) \to \mathrm{H}^{p+1}(k, F')$ *for* $p = 0$, 1.

c) *The Corestriction Principle for images holds for* $\mathrm{H}^p(k, \bar{G}) \to \mathrm{H}^{p+1}(k, \tilde{F})$, *for* $p = 0$, 1.

d) *The Corestriction Principle for images holds for* $ab_G^p : \mathrm{H}^p(k, G) \to \mathrm{H}^p_{ab}(k, G)$ *for any such* G.

For the statements a) − d) considered above, let us denote by $x(p, q)$ the statement x) evaluated at (p, q), for $0 \leq p \leq q \leq 2$. For example, $a(1, 2)$ means the statement a) with $p = 1$, $q = 2$.
We will show later that if one of these conditions holds (e.g. if k is a local or global field) then for any isogeny of connected reductive k-groups $1 \to F \to G_1 \to G_2 \to 1$, the Corestriction Principle for the image of $\mathrm{H}^p(k, G_2) \to \mathrm{H}^{p+1}(k, F)$, $p = 0, 1$ holds.

We have the following results.

2.1. Theorem. 1) *All statements a) − d) are equivalent.*
2) *We have the following interdependence between the statements a) − d) with particular values of* p *and* q.

a) *For lower dimension :*

$$a(0, 0) \Leftrightarrow b(0) \Leftrightarrow c(0) \Leftrightarrow d(0)$$
$$\Downarrow$$
$$a(0, 1)$$

b) *For higher dimension :*

$$a(1, 1) \Leftrightarrow b(1) \Leftrightarrow c(1) \Leftrightarrow d(1)$$
$$\Downarrow$$
$$a(1, 2)$$

where two statements in the same row are connected by $\Leftrightarrow$ *if they are equivalent and the down arrow indicates that the statements standing below follow the ones standing above.*

We just indicate the logical dependence and the scheme of the proof of the statements of 1) : *d)* $\Rightarrow$ *a)* ; *a)* $\Rightarrow$ *d)* ; *b)* $\Leftrightarrow$ *c)* ; *c)* $\Leftrightarrow$ *a)* ; *a)* $\Rightarrow$ *b)*.

From the proofs of propositions above we derive several consequences.

2.2. Corollary. *If either one of the conditions a) or d) holds (e.g. if* k *is a local or global field of characteristic 0) then for any isogeny of connected reductive* k*-groups*

$$1 \to F \to G_1 \to G_2 \to 1,$$

the Corestriction Principle for images holds for standards maps

$$\mathrm{H}^p(k, G_2) \to \mathrm{H}^{p+1}(k, F),\ p = 0, 1.$$

2.3. Remarks. 1) From the proof of Theorem 1.2, its corollary and Theorem 2.1 one may deduce a new proof of Deligne's result mentioned above ([5], Prop. 2.4.8) in the case k is a local or global field of characteristic 0.

Corollary. *If k is a local or global field of characteristic* 0 *then* d (0) *holds. In particular Theorem* 1.1 *holds.*

2) A known sufficient condition for $c(0)$ to hold is that the group of R-equivalence of G over k' is trivial, i.e., $G(k')/R = 1$, since the Norm Theorem for the group of elements R-equivalent to 1 holds (see [6], Prop. 3.3.2). In [11], Theorem 1, Merkurjev proved, among other results, a Norm Theorem from which the above result of [6] follows.

3) The proof of Theorem 2.1 reduces the proof of Corestriction Principle for images for connected reductive groups to that of the maps

$$\mathrm{H}^p(k, \bar{G}) \to \mathrm{H}^{p+1}(k, \tilde{F}),$$

where $\tilde{F}$ is the center of a simply connected semisimple k-group $\tilde{G}$ with adjoint group $\bar{G}$. It is clear that we can reduce further to the case where $\bar{G}$ is almost simple. In this case, the Corestriction Principle for images is known for the case ${}^1A_n$, B_n (due to the rationality of $\bar{G}$ and the result of Gille-Merkurjev mentioned above), C_n([20]).

3. Corestriction Principle for R-equivalence groups. Let G be a k-group. Two points $x, y \in G(k)$ are called strictly R-*equivalent* (after Manin) if there is a map $f: \mathbf{P}^1 \to G$ defined over k and regular at 0 and 1, such that $f(0) = x$ and $f(1) = y$ (see [4] for more detalis). The equivalence relation generated from this is called R- *equivalence.* The subset $R := RG(k)$ of all elements of $G(k)$ which are R-equivalent to the identity is a *normal subgroup* of $G(k)$. It is well-known (see [4]) that for a field k of characteristic 0, the factor group $G(k)/R$, called *the group of R-equivalence classes* of G over k, is a birational invariant of the group G. In general, the study of the group $G(k)/R$ provides interesting information about the arithmetico-group-theoretic structure of the group $G(k)$, especially because there are many (even semisimple) groups with non-trivial R-equivalence groups (even over number fields).

In this section we are interested in the Corestriction Principle for images for $G(k)/R$ over local and global fields of characteristic 0. We use the notion of standard maps introduced in the introduction. By using our previous results and a result of [6], we obtain.

3.1. Theorem. *Assume that k is a local or global field of characteristic* 0. *Then for any connected reductive k-group G, a k-torus T, a standard map $\pi: G(k) \to T(k)$ and for any finite extension k' of k, the norm homomorphism $T(k') \to T(k)$ induces a canonical functorial norm map for images*

$$N_{k'/k}: \mathrm{Im}\,(G(k')/R \to T(k')/R) \to \mathrm{Im}\,(G(k)/R \to T(k)/R).$$

3.2. Corollary. *With above notation, for any isogeny of connected k-groups*

$$1 \to F \to H \to G \to 1,$$

with finite F, the Corestriction Principle for images holds for the map

$$G(k)/R \to (\mathrm{Im}\,(\delta))/R,$$

where δ is the connecting map $G(k) \to \mathrm{H}^1(k, F)$, and the R-equivalence in $Im(\delta)$ is induced from that of $G(k)$ as defined in [6].

The proof uses Ono's crossed diagram as in the course of proving Theorem 2.1.

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