Generating functions of the Jacobi polynomials and related Hilbert spaces of analytic functions

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1. Introduction. In the previous paper [5], we showed that a generating function of the Gegenbauer polynomials can be regarded as the integral kernel of a unitary mapping from an L^2 space onto a Hilbert space of analytic functions. Moreover, we gave in [6] a similar construction for the system of the zonal spherical functions on the homogeneous space $U(n)/U(n-1)$, which is geometrically analogous to the space $SO(n)/SO(n-1)$ whose zonal spherical functions are essentially given by the Gegenbauer polynomials. Problems of this kind were discussed first in [1]. The purpose of this paper is to show that a similar construction is also possible for the Jacobi polynomials, which are generalizations of the Gegenbauer polynomials.

Let $\boldsymbol{R}$, $\boldsymbol{C}$ be the fields of real and complex numbers, respectively. For positive numbers α and β, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $n = 0, 1, 2, \cdots$, are defined by the Rodrigues formula (cf. [2]):

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}].$$

Then the system $\{P_n^{(\alpha,\beta)}(x)\,;\, n = 0, 1, 2, \cdots\}$ has the orthogonality relations (cf. [2]):

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = \begin{cases} 0 & (n \neq m) \\ \dfrac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\dfrac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} & (n = m), \end{cases}$$

and the generating function (cf. [4]): for $-1 < x < 1$ and $z \in \boldsymbol{C}$, $|z| < 1$,

$$\sum_{n=0}^{\infty} \frac{(2n+\alpha+\beta+1)(\alpha+\beta+1)_n}{(\alpha+1)_n} z^n P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+1)(z+1)}{(1-z)^{\alpha+\beta+2}}\, {}_2F_1\left(\frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2}; \alpha+1; \frac{2z(x-1)}{(1-z)^2}\right),$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ (Γ is the Gamma function) and ${}_2F_1(a, b; c; t)$ is the Gaussian hypergeometric function. We denote by $F_{\alpha,\beta}(z, x)$ the right hand side of this formula.

Let $\varphi_n^{(\alpha,\beta)}(x)$ be the normalization of $P_n^{(\alpha,\beta)}(x)$ with respect to the inner product defined by

$$(\psi, \varphi)_{\alpha,\beta} = \int_{-1}^{1} \overline{\psi(x)}\varphi(x)(1-x)^\alpha(1+x)^\beta dx.$$

Then the system of the functions $\varphi_n^{(\alpha,\beta)}(x)$, $n = 0, 1, 2, \cdots$, is an orthonormal basis of the Hilbert space $\mathscr{L}^2_{\alpha,\beta} = L^2\left((-1, 1), (1-x)^\alpha(1+x)^\beta\right)$ with the inner product $(\ ,\ )_{\alpha,\beta}$.

In this paper, we shall give a Hilbert space $\mathscr{H}_{\alpha,\beta}$ of analytic functions and a unitary operator of $\mathscr{L}^2_{\alpha,\beta}$ onto $\mathscr{H}_{\alpha,\beta}$ whose integral kernel is the generating function $F_{\alpha,\beta}(z, x)$.

Suppose that α, β are positive numbers throughout this paper.

2. Hilbert space $\mathscr{H}_{\alpha,\beta}$. We define the function $\rho_{\alpha,\beta}(t)$ for $0 < t < 1$ by

$$\rho_{\alpha,\beta}(t) = t^{\frac{\alpha+\beta-1}{2}} \int_t^1 u^{-\frac{\alpha+\beta+1}{2}}(1-u)^{\beta-1}du \int_{\frac{t}{u}}^1 v^{-\frac{\beta-\alpha+1}{2}}(1-v)^{\beta-1}dv,$$

and denote by $\mathscr{H}_{\alpha,\beta}$ the Hilbert space of analytic functions on the unit open disk B in $\boldsymbol{C}$ with the inner product defined by

$$\langle f, g \rangle_{\alpha,\beta} = \int_B \overline{f(z)}g(z)\rho_{\alpha,\beta}(|z|^2)dz,$$

where $dz = dxdy$, $z = x + iy$ ($x, y \in \boldsymbol{R}$). The functions $g_n(z) = z^n$, $n = 0, 1, 2, \cdots$, form an orthogonal basis in $\mathscr{H}_{\alpha,\beta}$ and the norm $\|g_n\| = \sqrt{\langle g_n, g_n \rangle_{\alpha,\beta}}$ is given in the following.

Lemma 1. *For a nonnegative integer n, we have*

$$\langle g_n, g_n \rangle_{\alpha,\beta} = \frac{2\pi\left(\Gamma(\beta)\right)^2}{2n+\alpha+\beta+1}\frac{\Gamma(n+1)}{\Gamma(n+\beta+1)}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+1)}$$

Proof. In exchanging orders of integrals, we obtain,

$$
\begin{aligned}
< g_n, g_n >_{\alpha,\beta} &= \pi\int_0^1 t^n \rho_{\alpha,\beta}(t)dt \\
&= \pi\int_0^1 t^{n+\frac{\alpha+\beta-1}{2}}dt\int_t^1 u^{-\frac{\alpha+\beta+1}{2}}(1-u)^{\beta-1}du\int_{\frac{t}{u}}^1 v^{-\frac{\beta-\alpha+1}{2}}(1-v)^{\beta-1}dv \\
&= \pi\int_{u=0}^1\int_{v=0}^1 (1-u)^{\beta-1}v^{\alpha}(1-v)^{\beta-1}\left[(uv)^{-\frac{\alpha+\beta+1}{2}}\int_0^{uv} t^{n+\frac{\alpha+\beta-1}{2}}dt\right]dudv \\
&= \frac{2\pi}{2n+\alpha+\beta+1}\int_{u=0}^1\int_{v=0}^1 (1-u)^{\beta-1}v^{\alpha}(1-v)^{\beta-1}(uv)^n dudv \\
&= \frac{2\pi}{2n+\alpha+\beta+1}\frac{\Gamma(n+1)\Gamma(\beta)}{\Gamma(n+\beta+1)}\frac{\Gamma(n+\alpha+1)\Gamma(\beta)}{\Gamma(n+\alpha+\beta+1)},
\end{aligned}
$$

which implies our assertion.

Remark 1. *The function $\rho_{\alpha,\beta}(t)$ has also the following expression.*

$$
\rho_{\alpha,\beta}(t) = t^{\frac{\alpha+\beta-1}{2}}\iint_{\substack{t\le u,v\le 1\\ uv\ge t}} u^{-\frac{\alpha+\beta+1}{2}}(1-u)^{\beta-1}v^{-\frac{\beta-\alpha+1}{2}}(1-v)^{\beta-1}dudv.
$$

The integral of the right hand side has some analo gy to the incomplete beta functions.

3. Main theorem. Let $u_n^{(\alpha,\beta)}(z)$ be the normalization of $g_n(z)$ with respect to the inner product of $\mathscr{H}_{\alpha,\beta}$. Then the system of the functions $u_n^{(\alpha,\beta)}(z)$, $n = 0, 1, 2, \cdots$, is an orthonormal basis in $\mathscr{H}_{\alpha,\beta}$.

It is indeed easy to see that,

$$
\begin{aligned}
&\sum_{n=0}^{\infty} u_n^{(\alpha,\beta)}(z)\,\varphi_n^{(\alpha,\beta)}(x) \\
&= \sum_{n=0}^{\infty}\frac{2n+\alpha+\beta+1}{\sqrt{2^{\alpha+\beta+2}\pi}}\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(\beta)}z^nP_n^{(\alpha,\beta)}(x) \\
&= \frac{\Gamma(\alpha+\beta+1)}{\sqrt{2^{\alpha+\beta+2}\pi}\,\Gamma(\alpha+1)\Gamma(\beta)} \\
&\quad\times\sum_{n=0}^{\infty}\frac{(2n+\alpha+\beta+1)(\alpha+\beta+1)_n}{(\alpha+1)_n}z^nP_n^{(\alpha,\beta)}(x) \\
&= \frac{\Gamma(\alpha+\beta+1)}{\sqrt{2^{\alpha+\beta+2}\pi}\,\Gamma(\alpha+1)\Gamma(\beta)}F_{\alpha,\beta}(z, x).
\end{aligned}
$$

We shall denote the last expression by $A_{\alpha,\beta}(z, x)$.

Theorem 1. *A unitary operator, $f = A_{\alpha,\beta}(\varphi)$, of $\mathscr{L}^2_{\alpha,\beta}$ onto $\mathscr{H}_{\alpha,\beta}$ is defined by*

$$
f(z) = \int_{-1}^1 A_{\alpha,\beta}(z, x)\varphi(x)(1-x)^{\alpha}(1+x)^{\beta}dx.
$$

Proof. For any $z\in B$, we have

$$
\sum_{n=0}^{\infty}\left|u_n^{(\alpha,\beta)}(z)\right|^2 < \infty.
$$

So we can consider that the series

$$
\sum_{n=0}^{\infty} u_n^{(\alpha,\beta)}(z)\varphi_n^{(\alpha,\beta)}(x)
$$

is the Fourier expansion for $A_{\alpha,\beta}(z, x)$ as a function of x. Thus, for $\varphi\in\mathscr{L}^2_{\alpha,\beta}$, we have

$$
\begin{aligned}
\Big(A_{\alpha,\beta}(\varphi)\Big)(z) &= \left(\sum_{n=0}^{\infty} u_n^{(\alpha,\beta)}(\bar{z})\,\varphi_n^{(\alpha,\beta)},\ \varphi\right)_{\alpha,\beta} \\
&= \sum_{n=0}^{\infty}(\varphi_n^{(\alpha,\beta)},\ \varphi)_{\alpha,\beta}\cdot u_n^{(\alpha,\beta)}(z),
\end{aligned}
$$

which gives the Fourier expansion for the function $\Big(A_{\alpha,\beta}(\varphi)\Big)(z)$. Hence, we obtain

$$
\begin{aligned}
&< A_{\alpha,\beta}(\varphi), A_{\alpha,\beta}(\varphi) >_{\alpha,\beta} \\
&= \sum_{n=0}^{\infty}\left|(\varphi_n^{(\alpha,\beta)},\ \varphi)_{\alpha,\beta}\right|^2 = (\varphi,\ \varphi)_{\alpha,\beta}.
\end{aligned}
$$

This implies that the operator $A_{\alpha,\beta}$ is unitary. Moreover, $A_{\alpha,\beta}(\varphi_n^{(\alpha,\beta)}) = u_n^{(\alpha,\beta)}$, which means $A_{\alpha,\beta}$ is surjective. So we can conclude that the assertions are true.

Remark 2. *The Gegenbauer polynomials $C_n^{\lambda}(x)$, $n = 0, 1, 2, \cdots$, are defined as the Jacobi polynomials with $\alpha = \beta = \lambda - \frac{1}{2}$ (cf. [2]):*

$$
C_n^{\lambda}(x) = \frac{(2\lambda)_n}{\left(\lambda+\frac{1}{2}\right)_n}P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x).
$$

The functions $\rho_{\alpha,\beta}(t)$, $A_{\alpha,\beta}(z, x)$ for $\alpha = \beta = \lambda - \frac{1}{2} > 0$ are given as follows:

$$
\rho_{\alpha,\beta}(t) = \frac{\left(\Gamma(\lambda-\frac{1}{2})\right)^2}{\Gamma(2\lambda-1)}t^{\lambda-1}\int_t^1 s^{-\lambda}(1-s)^{2\lambda-2}ds,
$$

and

$$
\begin{aligned}
&A_{\alpha,\beta}(z, x) \\
&= \frac{\Gamma(2\lambda+1)}{2^{\lambda+\frac{1}{2}}\sqrt{\pi}\,\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(\lambda-\frac{1}{2}\right)}\frac{1-z^2}{(1-2xz+z^2)^{\lambda+1}},
\end{aligned}
$$

which differ only by constant multiples from the conclusions in [5].

References

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