A remark on 2-microhyperbolicity

By Otto LIESS,[*)] Yasunori OKADA,[**)] and Nobuyuki TOSE[***)]

(Communicated by Kiyosi ITÔ, M. J. A., March 12, 1998)

Introduction. The notion of microhyperbolicity due to Kashiwara-Kawai [3] has played an important role in many problems concerning existence and regularity of solutions of linear PDE in microfunctions. When we study microfunction solutions from a second microlocal point of view, new phenomena appear. In fact, the basic notion is that of the second wave front set, and this notion is already for opportunity reasons closely related to that of second-hyperfunctions. respectively second-microfunctions. (We shall call them 2-hyperfunctions and 2-microfunctions henceforth). It is then natural to introduce the notion of 2-microhyperbolicity, a notion which turns out to be quite efficient for regularity problems (cf. N. Tose [11]) but which seems to have no immediate applications for existence problems in standard microfunctions, in view of the fact that the space of 2-hyperfunctions is much larger than is the space of microfunctions.

The main purpose of the present paper is to clarify the situation by examining an example which is modelled on the classical Mizohata operator. Indeed, this operator shall be 2-microhyperbolic (with respect to some regular involutive homogeneous submanifold in the phase space), it will be solvable in 2-hyperfunctions, but we shall see that it is not solvable in standard microfunctions (see Theorem 2.1 below). Since we think that the primary object of study should always be the "equation" and since we "believe" in the importance of the notion of 2-microhyperbolicity, we expect that this example gives an additional argument in favour of 2-hyperfunctions as the correct frame in which one should perform 2-microlocal arguments.

1. Preliminaries. 1.1. 2-microlocal analysis. Since the problem is microlocal, we take, from the beginning, a local model V of a regular involutive homogeneous submanifold in the cotangent bundle $\sqrt{-1}\,T^*\mathbf{R}^n$ defined by

(1) $V := \{(x; \sqrt{-1}\xi\cdot dx) \in \sqrt{-1}T^*\mathbf{R}^n;\ \xi_1 = \cdots = \xi_d = 0\}.$

Here $x = (x_1, \dots, x_n)$ is a system of coordinates in $\mathbf{R}^n$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$ are the dual coordinates. We take the complexification of V in $T^*\mathbf{C}^n$ defined by

(2) $V^{\mathbf{C}} := \{(z; \zeta\cdot dz) \in T^*\mathbf{C}^n;\ \zeta_1 = \cdots = \zeta_d = 0\}.$

where $z = (z_1, \dots, z_n)$ is a complex coordinate system of $\mathbf{C}^n$ corresponding to x and $\zeta = (\zeta_1, \dots, \zeta_n)$ are the dual coordinates. Then a partial complexification $\tilde{V}$ of V in $V^{\mathbf{C}}$ is given by

(3) $\tilde{V} := \{(z; \zeta\cdot dz) \in T^*\mathbf{C}^n;\ \zeta_1 = \cdots = \zeta_d = 0,\ \Im z_{d+1} = \cdots = \Im z_n = 0,\ \Re\zeta_{d+1} = \cdots = \Re\zeta_n = 0\}.$

This space can be identified with the conormal bundle $T_N^*\mathbf{C}^n$ of

(4) $N := \{z \in \mathbf{C}^n;\ \Im z_{d+1} = \cdots = \Im z_n = 0\}.$

The space $\tilde{V}$ is endowed with the sheaf $\mathscr{C}_{\tilde{V}}$ of microfunctions with holomorphic parameters $z' = (z_1, \dots, z_d)$ which is defined by

(5) $$\mathscr{C}_{\tilde{V}} = H^{n-d}(\mu_N(\mathscr{O}_{\mathbf{C}^n}))$$

by means of the Sato's microlocalization functor along N (refer to Kashiwara-Schapira [4] for this). First remark that the sheaf $\mathscr{A}_V^2 := \mathscr{C}_{\tilde{V}}|_V$ is a subsheaf of the sheaf $\mathscr{C}_{\mathbf{R}^n}$ of microfunctions on $\mathbf{R}^n$. Thus we have an exact sequence

(6) $$0 \to \mathscr{A}_V^2 \to \mathscr{C}_{\mathbf{R}^n}|_V$$

on V. To analyze the gap between the two sheaves, M. Kashiwara introduced the sheaf $\mathscr{C}_V^2$ of 2-microfunctions along V on $T_V^*\tilde{V}$ by

(7) $$\mathscr{C}_V^2 := H^d(\mu_V(\mathscr{C}_{\tilde{V}})).$$

The sheaf $\mathscr{C}_V^2$ gives rise to the exact sequences

(8) $$0 \to \mathscr{C}_{\mathbf{R}^n}|_V \to \mathscr{B}_V^2,$$

and

(9) $$0 \to \mathscr{A}_V^2 \to \mathscr{B}_V^2 \to \mathring{\pi}_{V*}(\mathscr{C}_V^2) \to 0.$$

Here the sheaf $\mathscr{C}_V^2$ restricted to the zero-section V of $T_V^*\tilde{V}$

(10) $$\mathscr{B}_V^2 := \mathscr{C}_V^2|_V$$

is the sheaf of 2-hyperfunctions, and

$$\mathring{\pi}_V : \mathring{T}_V^*\tilde{V} := T_V^*\tilde{V} \setminus V \to V$$

is the natural projection. It should be noted that the morphism $\mathscr{C}_{\mathbf{R}^n}|_V \to \mathscr{B}_V^2$ is not surjective, and we refer to Kataoka-Okada-Tose [5] for an expli-

*) Department of Mathematics, University of Bologna, Italy.
**) Faculty of Science, Chiba University.
***) Faculty of Economics, Keio University.

cit example of why this is the case.

Moreover we have the 2-spectral morphism

(11) $$Sp_V^2 : \pi_V^{-1}\mathscr{B}_V^2 \to \mathscr{C}_V^2$$

(with $\pi_V : T_V^*\tilde{V} \to V$). By means of Sp_V^2, we define the 2-singular spectrum $SS_V^2(u)$ along V of a microfunction $u \in \mathscr{C}_{\mathbf{R}^n}|_V$ by

(12) $$SS_V^2(u) := \mathrm{supp}(Sp_V^2(u)).$$

In view of (9), $SS_V^2(u)$ gives rather precise informations on the regularity of u if one studies propagation of 2-microlocal singularities. On the contrary, the sheaf $\mathscr{C}_V^2$ does not seem to be the adequate tool to study solvability of linear PDE in the frame of classical hyperfunctions. To take this into account, Kataoka-Okada-Tose [5] have constructed a sheaf $\tilde{\mathscr{C}}_V^2$ on $\mathring{T}_V^*\tilde{V}$ with the exact sequence

(13) $$0 \to \mathscr{A}_V^2 \to \mathscr{C}_{\mathbf{R}^n}|_V \to \mathring{\pi}_{V*}(\tilde{\mathscr{C}}_V^2) \to 0$$

1.2. 2-microhyperbolic directions We recall the notion of 2-microhyperbolic direction defined by N. Tose [11] using the notations in §1.1.

Let $\mathscr{M}$ be a coherent $\mathscr{E}_{\mathbf{C}^n}$ module. Then a co-direction $\theta \in T^*T_V^*\tilde{V}$ is 2-microhyperbolic along V for $\mathscr{M}$ in the sense of N. Tose [11] if the characteristic variety $char(\mathscr{M})$ of $\mathscr{M}$ satisfies the condition

(14) $$\theta \notin C_{T_V^*\tilde{V}}(C_{\tilde{V}}(char(\mathscr{M}))).$$

Here $C_{T_V^*\tilde{V}}(\cdot)$ (resp. $C_{\tilde{V}}(\cdot)$) is the normal cone along $T_V^*\tilde{V}$ (resp. $\tilde{V}$), which is a subset in $T_{T_V^*\tilde{V}}T^*\tilde{V}$ (resp. $T_{\tilde{V}}T^*C^n$). Moreover we consider the identifications

(15) $$(-H)^{-1} : T_{T_V^*\tilde{V}}T^*\tilde{V} \to T^*T_V^*\tilde{V}$$

and

(16) $$(-H)^{-1} : T_{\tilde{V}}T^*C^n \to T^*\tilde{V}$$

by using the Hamiltonian isomorphism H. Refer to Kashiwara-Schapira [4] for a detailed account of normal cones.

2. Main result. We follow the notation in §1. We assume, throughout §2, that $d = 2$ and that $n \geq 3$.

Let P be a microdifferential operator of order one of the form

(17) $$P = D_1 + \sqrt{-1}x_1D_2^2/D_n + P_0,$$

where P_0 is of 0-th order and P is defined in a neighborhood of $\mathring{q} = (0;\sqrt{-1}dx_n)$. P is thus a modification of the standard Mizohata operator $Q = D_1 + \sqrt{-1}\,x_1D_2$ and the solvability properties of P can in part be reduced to those of Q: see the argument later on.

First we study the operator P on the regular involutive submanifold

(18) $$V = \{(x;\sqrt{-1}\xi\cdot dx);\ \xi_1 = \xi_2 = 0\}.$$

Considered as a 2-microdifferential operator of Y. Laurent [6], P is 2-elliptic outside

(19) $$\Sigma = \{(x;\sqrt{-1}\,\xi''\cdot dx'';\sqrt{-1}\,(x_1^*dx_1 + x_2^*dx_2)) \in \mathring{T}_V^*\tilde{V};\ x_1^* = 0\},$$

where $(x;\sqrt{-1}\,\xi''\cdot dx'')$ denotes a point of V with $x'' = (x_3, \ldots, x_n)$. and $\xi'' = (\xi_3, \ldots, \xi_n)$. In fact, the 2-principal symbol of P is

(20) $$\sigma_{V^c}(P) = z_1^*$$

where z_j^* is the corresponding complex coordinate of x_j^* ($j = 1, 2$). It follows from the 2-ellipticity of P outside of Σ that

(21) $$P : \tilde{\mathscr{C}}_{V,q^*}^2 \to \tilde{\mathscr{C}}_{V,q^*}^2$$

is surjective for $q^* \in \mathring{T}_V^*\tilde{V}\setminus\Sigma$. Moreover a result of N. Tose [12] shows that P is equivalent to D_1 at $q^* \in \Sigma$. This means that there exists an invertible 2-microdifferential operator $Q \in \mathscr{E}_{VC,q^*}^{2,\infty}$ of infinite order satisfying

(22) $$QPQ^{-1} = D_1,$$

which implies that (21) is surjective at $q^* \in \Sigma$.

Now let us recall the exact sequence

(23) $$0 \to \mathscr{A}_V^2 \to \mathscr{B}_V^2 \to \mathring{\pi}_{V*}\left(\mathscr{C}_V^2|_{\mathring{T}_V^*\tilde{V}}\right) \to 0$$

on V and take into account the surjectivity of

(24) $$P : \mathscr{A}_{V,q}^2 \to \mathscr{A}_{V,q}^2$$

at $q \in V$, which is deduced from the Cauchy-Kowalevsky type theorem for $\mathscr{A}_V^2$ solutions due to Bony-Schapira [1]. Then it follows that

(25) $$P : \mathscr{B}_{V,q}^2 \to \mathscr{B}_{V,q}^2$$

is surjective at any $q \in V$.

We show that for P, dx_1 is 2-microhyperbolic in the sense of N. Tose [11]. In fact, we have

(26) $$C_{\tilde{V}}(\cdot)_q \subset C_{V\mathbf{C}}(\cdot)_q$$

at $q \in \tilde{V}$, and

(27) $$C_{V\mathbf{C}}(char(P))_q = \{q^* \in \left(T_V^*\tilde{V}\right)_q;\ \sigma_{V\mathbf{C}}(P)(q^*) = 0\} = \{z_1^*(q^*) = 0\},$$

where we take a coordinate system of $\tilde{V}$ as $(z', x'';\sqrt{-1}\,\xi''\cdot dx'')$ with $z' = (z_1, z_2)$ and z_j^* is the dual variable of z_j ($j = 1, 2$). It is then immediate that

(28) $$dx_1 \notin C_{T_V^*\tilde{V}}(C_{\tilde{V}}(char(P))).$$

Next we study the operator

(29) $$P : \tilde{\mathscr{C}}_{V,q^*}^2 \to \tilde{\mathscr{C}}_{V,q^*}^2$$

for $q^* \in \mathring{T}_V^*\tilde{V}$, respectively the operator

(30) $$P : \mathscr{C}_{\mathbf{R}^n,q} \to \mathscr{C}_{\mathbf{R}^n,q}$$

for

(31) $$q \in \Gamma := \{q \in V;\ x_1(q) = 0,\ \xi_n(q) > 0\}.$$

A classical theorem due to M. Sato *et al.* [9] tells us that

(32) $$P : \mathscr{C}_{\mathbf{R}^n,q} \to \mathscr{C}_{\mathbf{R}^n,q}$$

is not surjective if

(33) $$q \in \{s \in \sqrt{-1}T^*\mathbf{R}^n ; \xi_1(s) = x_1(s) = 0, \xi_2(s) \neq 0, \xi_n(s) > 0\}.$$

Thus if $q \in \Gamma$, then

(34) $$q \in \operatorname{supp}(\mathscr{C}_{\mathbf{R}^n}/P\mathscr{C}_{\mathbf{R}^n}).$$

This implies that

(35) $$P : \mathscr{C}_{\mathbf{R}^n,q} \to \mathscr{C}_{\mathbf{R}^n,q}$$

is not surjective at $q \in \Gamma$ whatever q we choose.

Next we study (29). First we remark that

(36) $$P : \tilde{\mathscr{C}}^2_{V,q^*} \to \tilde{\mathscr{C}}^2_{V,q^*}$$

is surjective if

(37) $$q^* \notin \Sigma.$$

This can be proved in many ways. For example, express $f \in \tilde{\mathscr{C}}^2_{V,q^*}$ as a boundary value of holomorphic functions using a result of Okada-Tose [8]. Then we can apply Bony-Schapira s method [1], in particular Theorem 4.2.1 of [1], to solve the equation in the complex domain 2-microlocally. Another way is to construct a subring $\tilde{\mathscr{E}}^2_{V\mathbf{C}}$ of $\mathscr{E}^2_{V\mathbf{C}}$ enjoying the properties

(38) $$\tilde{\mathscr{E}}^2_{V\mathbf{C}} \text{ acts on } \tilde{\mathscr{C}}^2_V$$

and

(39) $$\text{if } \sigma_{V\mathbf{C}}(P) \neq 0,\ P \in \tilde{\mathscr{E}}^2_{V\mathbf{C}} \Rightarrow P^{-1} \in \tilde{\mathscr{E}}^2_{V\mathbf{C}}.$$

This subring is implicity constructed in O. Liess [7].

Next we study the case

(40) $$q^* \in \Sigma_0 = \{q^* \in \Sigma ; x_1(q^*) = 0\}.$$

We remark that Σ_0 consists of two connected components

(41) $$\Sigma_0^+ = \{(q ; \sqrt{-1}x_2^* dx_2) \in \Sigma_0 ; x_2^* > 0, q \in \Gamma\}$$

and

(42) $$\Sigma_0^- = \{(q ; \sqrt{-1}x_2^* dx_2) \in \Sigma_0 ; x_2^* < 0, q \in \Gamma\}.$$

It follows from the exact sequence

(43) $$0 \to \mathscr{A}^2_V \to \mathscr{C}_{\mathbf{R}^n}|_V \to \mathring{\pi}_{V*}\left(\tilde{\mathscr{C}}^2_V|_{\mathring{T}^*_V\tilde{V}}\right) \to 0$$

that the coherent $\mathscr{E}_{\mathbf{C}^n}$ module $\mathscr{M} := \mathscr{E}_{\mathbf{C}^n}/\mathscr{E}_{\mathbf{C}^n}P$ enjoys an isomorphism

(44) $$\operatorname{Ext}^1_{\mathscr{E}_{\mathbf{C}^n}}(\mathscr{M}, \mathscr{C}_{\mathbf{R}^n,q}) \simeq \operatorname{Ext}^1_{\mathscr{E}_{\mathbf{C}^n}}(\mathscr{M}, \tilde{\mathscr{C}}^2_{V,(q;\sqrt{-1}dx_2)}) \oplus \operatorname{Ext}^1_{\mathscr{E}_{\mathbf{C}^n}}(\mathscr{M}, \tilde{\mathscr{C}}^2_{V,(q;-\sqrt{-1}dx_2)})$$

at $q \in V$. (Remark that, as is explained in studying $\mathscr{B}^2_V$ solutions,

(45) $$\operatorname{Ext}^1_{\mathscr{E}_{\mathbf{C}^n}}(\mathscr{M}, \mathscr{A}^2_{V,q}) = 0).$$

Thus at least at one point q^* of the set

(46) $$\{(q ; \sqrt{-1}dx_2), (q ; -\sqrt{-1}dx_2)\},$$

the morphism

(47) $$P : \tilde{\mathscr{C}}^2_{V,q^*} \to \tilde{\mathscr{C}}^2_{V,q^*}$$

is not surjective. We also note that P has simple characteristics on *char* $(\mathscr{M})$. Thus we have an isomorphism

(48) $$\mathscr{E}_{\mathbf{C}^n}/\mathscr{E}_{\mathbf{C}^n}P \simeq \mathscr{E}_{\mathbf{C}^n}/\mathscr{E}_{\mathbf{C}^n}P_1$$

where P_1 is the principal part of P:

(49) $$P_1 = D_1 + \sqrt{-1}x_1 D_2^2/D_n.$$

Since P_1 is invariant under the coordinate change

(50) $$x_2 \to -x_2,$$

we have an isomorphism

(51) $$\operatorname{Ext}^1_{\mathscr{E}_{\mathbf{C}^n}}(\mathscr{M}, \tilde{\mathscr{C}}^2_{V,(q;\sqrt{-1}dx_2)}) \simeq \operatorname{Ext}^1_{\mathscr{E}_{\mathbf{C}^n}}(\mathscr{M}, \tilde{\mathscr{C}}^2_{V,(q;-\sqrt{-1}dx_2)}).$$

Accordingly, we have shown that

(52) $$\operatorname{Ext}^1_{\mathscr{E}_{\mathbf{C}^n}}(\mathscr{M}, \tilde{\mathscr{C}}^2_{V,q^*}) \neq 0$$

in both cases,

(53) $$q^* = (q ; \sqrt{-1}dx_2) \text{ and } q^* = (q ; -\sqrt{-1}dx_2).$$

We conclude that

(54) $$P : \tilde{\mathscr{C}}^2_{V,q^*} \to \tilde{\mathscr{C}}^2_{V,q^*}$$

is not surjective at any point q^* of

(55) $$\Sigma_0 = \{q^* \in \Sigma ; x_1(q^*) = 0\}.$$

We summarize what we have obtained in this section in

Theorem 2.1. *Let P be a microdifferential operator given in* (17). *Then we have*

(*i*) *the direction dx_1 is 2-microhyperbolic in the sense of N. Tose* [11].

(*ii*) *the operator*

(56) $$P : \tilde{\mathscr{C}}^2_{V,q^*} \to \tilde{\mathscr{C}}^2_{V,q^*}$$

is surjective for any $q^* \in \mathring{T}^*_V\tilde{V}$,

(*iii*) *the operator*

(57) $$P : \mathscr{B}^2_{V,q} \to \mathscr{B}^2_{V,q}$$

is surjective for any $q \in V$,

(*iv*) *the operator*

(58) $$P : \tilde{\mathscr{C}}^2_{V,q^*} \to \tilde{\mathscr{C}}^2_{V,q^*}$$

is not surjective at any

(59) $$q^* \in \Sigma_0 = \{q^* \in \Sigma ; x_1(q^*) = 0\},$$

(*v*) *the operator*

(60) $$P : \mathscr{C}_{\mathbf{R}^n,q} \to \mathscr{C}_{\mathbf{R}^n,q}$$

is not surjective if

(61) $$q \in \Gamma = \{q \in V ; x_1(q) = 0\}.$$

In particular, the equation $Pu = f$ is solvable in 2-hyperfunctions at any q, but is not always solvable within the frame of classical microfunctions.

Remark 2.2. Let W denote the characteristic variety of P. Then the regular involutive submanifold $W_0 = W \cap \{z_1 \neq 0\}$ in $T^*\mathbf{C}^n$ satisfies the condition that

(62) $$W_0 \cap \bar{W}_0$$

is a regular involutive submanifold of $T^*\mathbf{C}^n$ of codimension 2. Under this condition, we can find, thanks to [10], a real quantized contact transformation by which W_0 is transformed into the form

(63) $$\{\zeta_1 + \sqrt{-1}\zeta_2^2/\zeta_n = 0\}.$$

Moreover, since P is simple characteristic on W_0, P is transformed into

(64) $$P_1 = D_1 + \sqrt{-1}D_2^2/D_n.$$

This observation makes it possible to show that

(65) $$P : \mathscr{C}_{\mathbf{R}^n,q} \to \mathscr{C}_{\mathbf{R}^n,q}$$

is surjective if $q \in V$ satisfies

(66) $$x_1(q) \neq 0.$$

References

[1] J.-M. Bony and P. Schapira: Propagation des singularités analytiques pour les solutions des équations aux dérivés partielles. Ann. Inst. Fourier, **26 (1)**, 81–140 (1976).

[2] M. Kashiwara and Y. Laurent: Théorèmes d'annulation et deuxième microlocalisation. Orsay (1983) (Preprint).

[3] M. Kashiwara and T. Kawai: Microhyperbolic pseudo-differential operators I. J. Math. Soc. Japan, **27**, 359–404 (1975).

[4] M. Kashiwara and P. Schapira: Sheaves on Manifolds. Grundlehren der Math., **292**, Springer-Verlag, Berlin, Heidelberg (1990).

[5] K. Kataoka, Y. Okada, and N. Tose: Decomposition of second microlocal analytic singularities. D-modules and Microlocal Geometry (eds. M. Kashiwara *et al.*). Walter de Gruyter, Berlin, New York, pp. 163–171 (1992).

[6] Y. Laurent: Théorie de la deuxième microlocalisation dans le domaine complexe. Progress in Mathematics, **53**, Birkhäuser (1985).

[7] O. Liess: Conical Refraction and Higher Microlocalization. LN in Math., **1555**, Springer-Verlag, Berlin, Hidelberg (1993).

[8] Y. Okada and N. Tose: FBI transformation and second microlocalization. J. de Math. Pures et Appl., **70**, 427–453 (1991).

[9] M. Sato, T. Kawai, and M. Kashiwara: Hyperfunctions and pseudo-differential equations. LN in Math., **287**, Springer-Verlag, Berlin, Heidelberg, New York, pp. 265–429 (1973).

[10] M. Sato, T. Kawai, and M. Kashiwara: The theory of pseudodifferential equations in the theory of hyperfunctions. Sugaku, Japan Math. Soc., **25(3)**, 213–238 (1973).

[11] N. Tose: On a class of 2-microhyperbolic systems. J. de Math. Pures et Appl., **67**, 23–37 (1988).

[12] N. Tose: 2nd microlocalization and conical refraction. Algebraic Analysis; Volumes in Honor of Prof. M. Sato (eds. M. Kashiwara and T. Kawai). vol. 2, Academic Press, San Diego, pp. 867–882 (1988).