# On the $\mathbf{Z}_3$-extension of a certain cubic cyclic field

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In our previous paper [2], we gave the following Theorem for vanishing of Iwasawa invariants of a cyclic extension of odd prime degree over the rational number field $\boldsymbol{Q}$.

**Theorem A** ([2, Cor. 3.6.]). *Let $l$ be an odd prime number, $k$ a cyclic extension of degree $l$ over $\boldsymbol{Q}$, $\boldsymbol{Q}_\infty$ the cyclotomic $\boldsymbol{Z}_l$-extension of $\boldsymbol{Q}$ and $k_\infty = k\boldsymbol{Q}_\infty$ the composite field of $k$ and $\boldsymbol{Q}_\infty$. Then the following are equivalent:*

(1) *The Iwasawa $\lambda$-invariant $\lambda_l(k_\infty/k)$ of $k_\infty$ over $k$ is zero.*

(2) *For any prime ideal $\mathfrak{p}$ of $k_\infty$ which is prime to $l$ and ramified in $k_\infty$ over $\boldsymbol{Q}_\infty$, the order of the ideal class of $\mathfrak{p}$ is prime to $l$.*

Moreover, using Theorem A, we gave some examples of vanishing of $\lambda(k_\infty/k)$, in [2]. More precisely, let $\boldsymbol{Q}_1$ be the initial layer of the cyclotomic $\boldsymbol{Z}_3$-extension $\boldsymbol{Q}_\infty$ of $\boldsymbol{Q}$, $k$ a cubic cyclic extension over $\boldsymbol{Q}$ with prime conductor $p$ such that $p \equiv 1 \pmod 9$, $k_1 = k\boldsymbol{Q}_1$, $E_{\boldsymbol{Q}_1}$ (resp. $E_{k_1}$) the unit group of $\boldsymbol{Q}_1$ (resp. $k_1$) and $N_{k_1/\boldsymbol{Q}_1}$ the norm $k_1$ over $\boldsymbol{Q}_1$. In [2, Example 4.1], we treated the case $(E_{\boldsymbol{Q}_1} : N_{k_1/\boldsymbol{Q}_1}(E_{k_1})) = 9$ and $p \not\equiv 1 \pmod{27}$, which implies that the prime ideals of $k_1$ lying above $p$ are principal by genus formula. In this paper, we treat the case $p = 73$, which could not be treated in [2]. We note that if $p = 73$, then $(E_{\boldsymbol{Q}_1} : N_{k_1/\boldsymbol{Q}_1}(E_{k_1})) = 3$ (cf. [2, Example 4.2]).

The main purpose of this paper is to prove the following theorem:

**Theorem.** *Let $\zeta_{73} = e^{\frac{2\pi i}{73}}$, $k$ the unique subfield of $\boldsymbol{Q}(\zeta_{73})$ of degree 3 over $\boldsymbol{Q}$ and $k_\infty$ the cyclotomic $\boldsymbol{Z}_3$-extension of $k$. Then the $\lambda$-invariant $\lambda_3(k_\infty/k)$ of $k_\infty$ over $k$ is zero.*

The Theorem will be proved by using Fukuda's method (cf. [1]). We note that Leopoldt's conjecture is valid for the above $k$ (cf. [4, p. 71]) and $k$ is totally real. Now we explain notations.

We denote by $\boldsymbol{Z}$ the rational integer ring. We put $\zeta_n = e^{\frac{2\pi i}{n}}$ for a positive integer $n$. Let $F$ be a number field. We denote by $O_F$ the integer ring of $F$. For an integral ideal $\mathfrak{a}$ of $F$, we denote by $Cl(\mathfrak{a})$ the ideal class of $\mathfrak{a}$, $O_F/\mathfrak{a}$ the factor ring of $O_F$ over $\mathfrak{a}$ and $(O_F/\mathfrak{a})^\times$ the set of invertible elements of $O_F/\mathfrak{a}$. For a Galois extension $L$ of $F$, we denote by $G(L/F)$ the Galois group of $L$ over $F$. Let $G$ be a group. For elements $g_1, g_2, \dots, g_r$ of $G$, we denote by $\langle g_1, g_2, \dots, g_r\rangle$ the subgroup of $G$ generated by $g_1, g_2, \dots, g_r$.

In order to prove our Theorem, we shall use the following Lemma:

**Lemma 1** (cf. [3, Cor. of Prop. 1]). *Let $F$ be a totally real number field for which Leopoldt's conjecture is valid. Let $A_0$ be the $l$-sylow subgroup of the ideal class group of $F$ and $\mathfrak{a}$ a product of primes of $F$ lying above $l$ such that $Cl(\mathfrak{a}) \in A_0$. Then $\mathfrak{a}$ becomes principal in the $n$-th layer $F_n$ of $F_\infty$ over $F$ for sufficiently large $n$.*

Let $\boldsymbol{Q}_\infty$ be the cyclotomic $\boldsymbol{Z}_3$-extension of $\boldsymbol{Q}$ and $\boldsymbol{Q}_n$ the $n$-th layer of $\boldsymbol{Q}_\infty$ over $\boldsymbol{Q}$ for a non-negative integer $n$. We let $k_n = k\boldsymbol{Q}_n$ and $A_n$ the 3-sylow subgroup of the ideal class group of $k_n$. We put $\theta = \zeta_9 + \zeta_9^{-1} = 2\cos\frac{2\pi}{9}$. Then the roots of the equation $x^3 - 3x + 1 = 0$ are $\theta$, $\theta^2 - 2 = \zeta_9^7 + \zeta_9^{-7}$ and $-\theta^2 - \theta + 2 = \zeta_9^4 + \zeta_9^{-4}$. We note $\boldsymbol{Q}_1 = \boldsymbol{Q}(\theta)$ and $x^3 - 3x + 1 \equiv (x + 34)(x + 14)(x + 25) \pmod{73}$. Let $\mathfrak{p}_1$ be the ideal $(\theta + 34, 73)$ of $O_{\boldsymbol{Q}_1}$ generated by $\theta + 34$, 73. Since $N_{\boldsymbol{Q}_1/\boldsymbol{Q}}(\theta^2 + 6\theta - 3) = (\theta^2 + 6\theta - 3)(5\theta^2 - \theta - 11)(-6\theta^2 - 5\theta + 11) = -73$ and since $\theta^2 + 6\theta - 3 \equiv (\theta + 34)(\theta - 28) \pmod{73}$, we have $\mathfrak{p}_1 = (\theta^2 + 6\theta - 3)$. In a similar way, we have $(\theta + 14, 73) = (5\theta^2 - \theta - 11)$ and $(\theta + 25, 73) = (-6\theta^2 - 5\theta + 11)$. We put $\mathfrak{p}_2 = (5\theta^2 - \theta - 11)$ and $\mathfrak{p}_3 = (-6\theta^2 - 5\theta + 11)$. Note that $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are the distinct prime ideals of $\boldsymbol{Q}_1$ lying above 73 and $(O_{\boldsymbol{Q}_1}/\mathfrak{p}_i)^\times \cong (\boldsymbol{Z}/73\boldsymbol{Z})^\times$.

We put $P\mathfrak{m} = \{a \in \boldsymbol{Q}_1 ; a \text{ is prime to } \mathfrak{m}\}$ and $S\mathfrak{m} = \{a \in P\mathfrak{m} ; a \equiv 1 \pmod{\mathfrak{m}}\}$ for an ideal $\mathfrak{m}$ of $\boldsymbol{Q}_1$. Now, we define a surjective homomorphism $\varphi$ of $P_{73}/S_{73}$ to an abelian group $V =$

$\boldsymbol{Z}/3\boldsymbol{Z} \oplus \boldsymbol{Z}/3\boldsymbol{Z} \oplus \boldsymbol{Z}/3\boldsymbol{Z}$ as follows:

Since 5 mod 73 is a generator of a cyclic group $(\boldsymbol{Z}/73\boldsymbol{Z})^\times$, there exists an integer $e_a$ for any element $a$ mod $73 \in (\boldsymbol{Z}/73\boldsymbol{Z})^\times$ such that $(5 \bmod 73)^{e_a} = a \bmod 73$. Hence we can define asurjective homomorphism $\iota$ of $(\boldsymbol{Z}/73\boldsymbol{Z})^\times$ to $\boldsymbol{Z}/3\boldsymbol{Z}$ defined by $\iota(a \bmod 73) = e_a \bmod 3$. Then we can define the following surjective homomorphism $\varphi$ of $P_{73}/S_{73} (\cong (O_{Q_1}/\mathfrak{p}_1)^\times \times (O_{Q_1}/\mathfrak{p}_2)^\times \times (O_{Q_1}/\mathfrak{p}_3)^\times)$ to $V$ by $\varphi(f(\theta)) = (\iota(f(-34) \bmod 73), \iota(f(-14) \bmod 73), \iota(f(-25) \bmod 73))$, where $f(\theta)$ is a polynomial of $\theta$ with rational integral coefficients and $f(\theta) \in P_{73}$.

Now, let $K$ be the class field of $\boldsymbol{Q}_1$ corresponding to the subgroup $P_{73}^3 E_{Q_1} S_{73}$ of $P_{73}$, where $P_{73}^3 = \{\alpha^3 ; \alpha \in P_{73}\}$. Then, since the class number of $\boldsymbol{Q}_1$ is one, we have the isomorphism

$$\phi : P_{73}/P_{73}^3 E_{Q_1} S_{73} \ni \alpha P_{73}^3 E_{Q_1} S_{73} \mapsto \left(\frac{K/\boldsymbol{Q}_1}{(\alpha)}\right) \in G(K/\boldsymbol{Q}_1)$$

through Artin map.

Since $E_{Q_1}$ is the cyclotomic units of $\boldsymbol{Q}_1$ (cf. [4, p. 145]), $E_{Q_1}$ is generated by $\{-1, \zeta_9^{-\frac{1}{2}} \frac{1-\zeta_9^2}{1-\zeta_9} = -\theta^2 - \theta + 2, \zeta_9^{-\frac{3}{2}} \frac{1-\zeta_9^4}{1-\zeta_9} = -\theta^2 - \theta + 1\}$.

Now, for a simplicity, we denote by $(\bar{a}, \bar{b}, \bar{c})$ an element $(a \bmod 3, b \bmod 3, c \bmod 3) \in V$. Then we have $\varphi((-\theta^2 - \theta + 2) \bmod 73) = (\overline{-1}, \overline{-1}, \overline{-1})$ and $\varphi((-\theta^2 - \theta + 1)) = (\bar{1}, \bar{1}, \bar{1})$, which gives the isomorphism

$$\tilde{\varphi} : P_{73}/P_{73}^3 E_{Q_1} S_{73} \cong V/\langle(\overline{-1}, \overline{-1}, \overline{-1})\rangle$$

induced by $\varphi$.

Since $N_{Q_1/Q}(2-\theta) = 3$ and 3 mod 73 is a third power residue mod 73, the field $k_1 = k\boldsymbol{Q}_1$ is the class field of $\boldsymbol{Q}_1$ corresponding to $\langle 2 - \theta\rangle P_{73}^3 E_{Q_1} S_{73}$. This implies $\tilde{\varphi}\phi^{-1}(G(K/k_1)) = \langle(\bar{1}, \overline{-1}, \bar{0}), (\bar{1}, \bar{1}, \bar{1})\rangle/(\bar{1}, \bar{1}, \bar{1})\rangle$ by $\varphi(2-\theta) = (\bar{1}, \overline{-1}, \bar{0})$, which means $G(K/k_1) = \langle\left(\frac{K/\boldsymbol{Q}_1}{(2-\theta)}\right)\rangle$. We note that $K$ is 3-part of the genus field of $k_1$ over $\boldsymbol{Q}_1$ by class field theory, since $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are the prime ideals of $\boldsymbol{Q}_1$ which are ramified in $k_1$ over $\boldsymbol{Q}_1$.

**Lemma 2** (Ozaki). *Let $K'$ be the* 3-part of *the Hilbert class field of $k_1$ and $\mathfrak{L}$ a prime ideal of $k_1$ lying above* 3. *If $G(K'/k_1) = \langle\left(\frac{K'/k_1}{\mathfrak{L}}\right)\rangle$, then $\lambda_3(k_\infty/k) = 0$.*

*Proof.* Let $\mathfrak{P}_i$ be the prime ideal of $k_1$ lying above $\mathfrak{p}_i$. Since $\left(\frac{K'/k_1}{\mathfrak{P}}\right)$ is a power of $\left(\frac{K'/k_1}{\mathfrak{L}}\right)$, $\mathfrak{P}_i$ becomes principal in $k_\infty$ by Lemma 1. Moreover $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3$ are the prime ideals of $k_\infty$ which are ramified in $k_\infty$ over $\boldsymbol{Q}_\infty$, which shows $\lambda_3(k_\infty/k) = 0$ by Theorem A.

Since the ideal $(2-\theta)$ is the unique prime ideal of $\boldsymbol{Q}_1$ lying above 3, in order to prove our Theorem, it is sufficient to show $K = K'$ because of $\left(\frac{K'/k_1}{\mathfrak{L}}\right) = \left(\frac{K/k_1}{\mathfrak{L}}\right) = \left(\frac{K/\boldsymbol{Q}_1}{(2-\theta)}\right)$.

Let $k'$ be the class field of $\boldsymbol{Q}_1$ corresponding to $P_{\mathfrak{p}_2\mathfrak{p}_3}^3 E_{Q_1} S\mathfrak{p}_2\mathfrak{p}_3$. Then we have $\left(\frac{K/k_1}{\mathfrak{P}_1}\right)\Big|_{k'} = \left(\frac{k'/\boldsymbol{Q}_1}{\mathfrak{p}_1}\right)$ and hence $\left(\frac{K/k_1}{\mathfrak{P}_1}\right) = \left(\frac{K/k_1}{(2-\theta)}\right)$ by $\iota((\theta^2 + 6\theta - 3) \bmod \mathfrak{p}_2) = \bar{1}$ and $\iota((\theta^2 + 6\theta - 3) \bmod \mathfrak{p}_3) = \overline{-1}$. This implies $G(K/k_1) = \langle\left(\frac{K/k_1}{\mathfrak{P}_1}\right)\rangle$, which shows $K = K'$ by genus theory (cf. [5, Lemma 2]).

## References


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