On the vanishing of Iwasawa invariants of certain cyclic extensions of Q with prime degree II

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1. Introduction. Throughout the paper, we fix an odd prime number ℓ. For a prime number p congruent to one module ℓ, we denote by k_p the unique subfield of $\boldsymbol{Q}(\zeta_p)$ of degree ℓ, where ζ_p is a primitive p-th root of unity. Let $\boldsymbol{F}_\ell = \boldsymbol{Z}/\ell\boldsymbol{Z}$ and let $(\frac{a}{p})_\ell$ be the ℓ-th power residue symbol for an integer a. In [2], we proved the following theorem.

Theorem 1.1 (Corollary 2.3 in [2]). *Let p and q be distinct prime numbers congruent to one modulo ℓ satisfying $(\frac{\ell}{p})_\ell \neq 1$, $(\frac{p}{q})_\ell \neq 1$, $q \not\equiv 1 \pmod{\ell^2}$. Let $x, y, z \in \boldsymbol{F}_\ell$ such that $(\frac{q\ell^x}{p})_\ell = 1$, $(\frac{\ell p^y}{q})_\ell = 1$ and $pq^z \equiv 1 \pmod{\ell^2}$. If $xyz \neq -1$, then for any subfield k of k_pk_q of degree ℓ, the Iwasawa invariants $\lambda_\ell(k)$ and $\mu_\ell(k)$ are both zero.*

In this paper, we investigate the case $(\frac{p}{q})_\ell = 1$.

2. Theorems. Let p and q be distinct prime numbers congruent to one modulo ℓ. We assume that $p \not\equiv 1 \pmod{\ell^2}$, $q \not\equiv 1 \pmod{\ell^2}$, $(\frac{\ell}{p})_\ell \neq 1$ and $(\frac{q}{p})_\ell = (\frac{p}{q})_\ell = 1$. We treat the case $(\frac{\ell}{q})_\ell = 1$ and the case $(\frac{\ell}{q})_\ell \neq 1$ separately. In the case $(\frac{\ell}{q})_\ell = 1$, we have the following theorem.

Theorem 2.1. *Assume that $(\frac{\ell}{q})_\ell = 1$. Let k be a subfield of k_pk_q of degree ℓ which is different from k_p and k_q. If $p \notin E_k k^{\times\ell}$, then $\lambda_\ell(k)$ and $\mu_\ell(k)$ are both zero.*

Here E_k denotes the unit group of k. In the case $(\frac{\ell}{q})_\ell \neq 1$, we need to specify k explicitly. Let

$$\sigma = \left(\frac{k_p/\boldsymbol{Q}}{\ell}\right),\ \tau = \left(\frac{k_q/\boldsymbol{Q}}{\ell}\right)$$

be Frobenius automorphisms. We identify the Galois group $G(k_p/\boldsymbol{Q})$ with $G(k_pk_q/k_q)$ and $G(k_q/\boldsymbol{Q})$ with $G(k_pk_q/k_p)$ canonically. Then $G(k_pk_q/\boldsymbol{Q}) = <\sigma, \tau>$. If k is a subfield of k_pk_q with degree ℓ which is different from k_p and k_q, then $G(k_pk_q/k) = <\sigma\tau^i>$ for some $i \in \boldsymbol{F}_\ell^\times$. In this case, we have the following theorem.

Theorem 2.2. *Assume that $(\frac{\ell}{q})_\ell \neq 1$. Let k be a subfield of k_pk_q which corresponds to $<\sigma\tau^i>$ for some $i \in \boldsymbol{F}_\ell^\times$ and z the element of $\boldsymbol{F}_\ell^\times$ such that $pq^z \equiv 1 \pmod{\ell^2}$. If $pq^{z/i} \notin E_k k^{\times\ell}$, then $\lambda_\ell(k)$ and $\mu_\ell(k)$ are both zero.*

3. Proof. We shall prove Theorem 2.2. For a Galois extension k of $\boldsymbol{Q}$, we denote by $A(k)$ the ℓ-primary part of the ideal class group of k and $B(k)$ the subgroup of $A(k)$ consisting of elements which are invariant under the action of $G(k/\boldsymbol{Q})$. Let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_s$ be the prime ideals of k which are ramified in $k/\boldsymbol{Q}$. If $k/\boldsymbol{Q}$ is a cyclic extension of degree ℓ, then $B(k)$ is an ℓ-elementary abelian group of rank $s-1$ generated by $\mathrm{cl}(\mathfrak{p}_1), \mathrm{cl}(\mathfrak{p}_2), \ldots, \mathrm{cl}(\mathfrak{p}_s)$.

Let $\boldsymbol{Q}_1$ be the subfield of $\boldsymbol{Q}(\zeta_{\ell^2})$ of degree ℓ and put

$$\eta = \left(\frac{\boldsymbol{Q}_1/\boldsymbol{Q}}{q}\right).$$

Then $G(\boldsymbol{Q}_1/\boldsymbol{Q}) = <\eta>$. Let $\mathfrak{p}_p$ (resp. $\mathfrak{p}_q$) be the prime ideal of k lying over p (resp. q). Since $p \not\equiv 1 \pmod{\ell^2}$ and $q \not\equiv 1 \pmod{\ell^2}$, $\mathfrak{p}_p$ and $\mathfrak{p}_q$ inert in $k\boldsymbol{Q}_1/k$. So, if we show that both $\mathfrak{p}_p$ and $\mathfrak{p}_q$ become principal in $k\boldsymbol{Q}_1$, we have $\lambda_\ell(k) = \mu_\ell(k) = 0$ from Corollary 3.6 of [3].

In order to show that both $\mathfrak{p}_p$ and $\mathfrak{p}_q$ become

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principal in $k\boldsymbol{Q}_1$, we use a subfield F of $K = k_p k_q \boldsymbol{Q}_1$. We identify $G(k_p/\boldsymbol{Q})$ with $G(K/k_q\boldsymbol{Q}_1)$, $G(k_q/\boldsymbol{Q})$ with $G(K/k_p\boldsymbol{Q}_1)$ and $G(\boldsymbol{Q}_1/\boldsymbol{Q})$ with $G(K/k_pk_q)$ canonically. We consider σ, τ and η as elements of $G(K/\boldsymbol{Q})$. In this situation, the above k corresponds to $<\sigma\tau^i, \eta>$. Let F be a subfield of $k\boldsymbol{Q}_1$ of degree ℓ which is different from k and $\boldsymbol{Q}_1$. Such F corresponds to $<\sigma\tau^i, \sigma\eta^t>$ for some $t \in \boldsymbol{F}_\ell^\times$. Let $\mathfrak{P}_p$, $\mathfrak{P}_q$ and $\mathfrak{P}_\ell$ be the prime ideals of F lying over p, q and ℓ respectively. Since $\mathfrak{P}_p = \mathfrak{p}_p$, $\mathfrak{P}_q = \mathfrak{p}_q$ in $k\boldsymbol{Q}_1$ and $\mathfrak{P}_\ell$ becomes principal in $k\boldsymbol{Q}_1$, we see that if $\mathfrak{P}_p^a\mathfrak{P}_q^b\mathfrak{P}_\ell^c$ is principal in F for some integers a, b, c then $\mathfrak{p}_p^a\mathfrak{p}_q^b$ is principal in $k\boldsymbol{Q}_1$.

Now, since

$$\left(\frac{K/F}{\mathfrak{P}_p}\right)\Big|_{k_q} = \left(\frac{k_q/\boldsymbol{Q}}{p}\right) = 1 = \tau^0,$$

$$\left(\frac{K/F}{\mathfrak{P}_p}\right)\Big|_{Q_1} = \left(\frac{\boldsymbol{Q}_1/\boldsymbol{Q}}{p}\right) = \eta^{-z},$$

we have

$$\left(\frac{K/F}{\mathfrak{P}_p}\right) = (\sigma\tau^i)^0(\sigma\eta^t)^{-z/t} = \sigma^{-z/t}\eta^{-z}.$$

Similarly we have

$$\left(\frac{K/F}{\mathfrak{P}_q}\right) = (\sigma\tau^i)^{-1/t}(\sigma\eta^t)^{1/t} = \tau^{-i/t}\eta$$

and

$$\left(\frac{K/F}{\mathfrak{P}_\ell}\right) = (\sigma\tau^i)^{1/i}(\sigma\eta^t)^{1-1/i} = \sigma\tau\eta^{t-t/i}.$$

We identify $G(K/\boldsymbol{Q})$ with $\boldsymbol{F}_\ell^3$ by the correspondence

$$\sigma \leftrightarrow \begin{pmatrix}1\\0\\0\end{pmatrix}, \ \tau \leftrightarrow \begin{pmatrix}0\\1\\0\end{pmatrix}, \ \eta \leftrightarrow \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Then the matrix

$$M(i, t) = \begin{pmatrix} -z/t & 0 & 1 \\ 0 & -i/t & 1 \\ -z & 1 & t-t/i \end{pmatrix}$$

describes $\left(\frac{K/F}{\mathfrak{P}_p}\right)$, $\left(\frac{K/F}{\mathfrak{P}_q}\right)$ and $\left(\frac{K/F}{\mathfrak{P}_\ell}\right)$. Since the rank of $M(i, t)$ is two for all $i, t \in \boldsymbol{F}_\ell^\times$, we have $G(K/F) = <\left(\frac{K/F}{\mathfrak{P}_p}\right), \left(\frac{K/F}{\mathfrak{P}_q}\right), \left(\frac{K/F}{\mathfrak{P}_\ell}\right)>$. Furthermore, since K/F is an abelian unramified extension of degree ℓ^2, K is the ℓ-part of the genus field of F. Hence we have the following lemma from Corollary 1.2 in [2].

Lemma 3.1. *For any subfield F of K of degree ℓ in which p, q and ℓ are ramified, we have $A(F) = B(F) \simeq \boldsymbol{F}_\ell^2$. Furthermore, for an ideal $\mathfrak{a}$ of F, $\mathfrak{a}$ is principal in F if and only if $\left(\frac{K/F}{\mathfrak{a}}\right) = 1$.*

From Lemma 3.1 we see immediately that $\mathfrak{P}_p\mathfrak{P}_q^{z/i}\mathfrak{P}_\ell^{z/t}$ is principal in F. Therefore $\mathfrak{p}_p\mathfrak{p}_q^{z/i}$ is principal in $k\boldsymbol{Q}_1$. Since $B(k) = <\mathrm{cl}(\mathfrak{p}_p), \mathrm{cl}(\mathfrak{p}_q)>$ is of order ℓ, there is a non-trivial relation between $\mathfrak{p}_p$ and $\mathfrak{p}_q$ in k. Hence, if $\mathfrak{p}_p\mathfrak{p}_q^{z/i}$ is not principal in k, both $\mathfrak{p}_p$ and $\mathfrak{p}_q$ become principal in $k\boldsymbol{Q}_1$. Since $\mathfrak{p}_p\mathfrak{p}_q^{z/i}$ is principal in k if and only if $pq^{z/i} \in E_k k^{\times\ell}$, we have proved Theorem 2.2.

The proof of Theorem 2.1 is similar. So we omit it.

4. Example. We give an example for $\ell = 3$. Readers are suggested to refer to [1] about cyclic cubic fields. Let $p = 7$ and $q = 223$. We are in the situation of Theorem 2.2. Then $pq^2 \equiv 1 \pmod 9$ and

$$\sigma = \left(\frac{k_p/\boldsymbol{Q}}{3}\right), \ \tau = \left(\frac{k_q/\boldsymbol{Q}}{3}\right).$$

Let θ_p be a root of $X^3 - X^2 - 2X + 1$. Then $k_p = \boldsymbol{Q}(\theta_p)$ and θ_p^σ is equal to $-1 - \theta_p + \theta_p^2$ or $2 - \theta_p^2$. Since 3 does not split in $k_p/\boldsymbol{Q}$, the Frobenius automorphism σ satisfies $\theta_p^\sigma \equiv \theta_p^3 \pmod 3$. So we have $\theta_p^\sigma = -1 - \theta_p + \theta_p^2$ and $\theta_p^{\sigma^2} = 2 - \theta_p^2 = 1 - \theta_p - \theta_p^\sigma$. Similarly if we let θ_q be a root of $X^2 - X^2 - 73X + 256$, then $k_q = \boldsymbol{Q}(\theta_q)$, $\theta_q^\tau = 26 - (5/2)\theta_q - (1/2)\theta_q^2$ and $\theta_q^{\tau^2} = 25 + (3/2)\theta_q + (1/2)\theta_q^2 = 1 - \theta_q - \theta_p^\tau$. Let k be a subfield of k_pk_q of degree 3 which is different from k_p and k_q. Then k corresponds to $<\sigma\tau>$ or $<\sigma\tau^2>$.

On the other hand, it is known that there are two cyclic cubic fields in which 3 and 227 are ramified. Such a field is the splitting field of $f_1(X) = X^3 - X^2 - 520X + 925$ or $f_2(X) = X^3 - X^2 - 520X - 2197$. We explain how to determine the polynomial corresponding to $<\sigma\tau>$.

Since $\{1, \theta_p, \theta_p^\sigma\}$ and $\{1, \theta_q, \theta_q^\tau\}$ are integral bases of k_p and k_q respectively and since the discriminants of k_p and k_q are relatively prime, $\{1, \theta_p, \theta_p^\sigma, \theta_q, \theta_p\theta_q, \theta_p^\sigma\theta_q, \theta_q^\tau, \theta_p\theta_q^\tau, \theta_p^\sigma\theta_q^\tau\}$ forms an integral basis of k_pk_q over $\boldsymbol{Z}$. We let $\boldsymbol{Z}^9$ be an $G(k_pk_q/\boldsymbol{Q})$ module via correspondence

$$(x_i) \leftrightarrow x_0 + x_1\theta_p + x_2\theta_p^\sigma + x_3\theta_q + x_4\theta_p\theta_q + x_5\theta_p^\sigma\theta_q + x_6\theta_q^\tau + x_7\theta_p\theta_q^\tau + x_8\theta_p^\sigma\theta_q^\tau.$$

Then the actions of σ and τ for $x = (x_i)$ are as follows:

$$x^{\sigma}=\begin{pmatrix} x_0+x_2\\ -x_2\\ x_1-x_2\\ -x_3+x_5\\ -x_5\\ x_4-x_5\\ x_6+x_8\\ -x_8\\ x_7-x_8 \end{pmatrix},\ x^{\tau}=\begin{pmatrix} x_0+x_6\\ x_1+x_7\\ x_2+x_8\\ -x_6\\ -x_7\\ -x_8\\ x_3-x_6\\ x_4-x_7\\ x_5-x_8 \end{pmatrix},\ x^{\sigma\tau}=\begin{pmatrix} x_0+x_2+x_6+x_8\\ -x_2-x_8\\ x_1-x_2+x_7-x_8\\ -x_6-x_8\\ x_8\\ -x_7+x_8\\ x_3+x_5-x_6-x_8\\ -x_5+x_8\\ x_4-x_5-x_7+x_8 \end{pmatrix}$$

Therefore, if we put

$$A=\begin{pmatrix} 1&0&1&0&0&0&1&0&1\\ 0&0&-1&0&0&0&0&0&-1\\ 0&1&-1&0&0&0&0&1&-1\\ 0&0&0&0&0&0&-1&0&-1\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&-1&1\\ 0&0&0&1&0&1&0&-1&-1\\ 0&0&0&0&0&-1&0&0&1\\ 0&0&0&0&1&-1&0&-1&1 \end{pmatrix},$$

then we have $x^{\sigma\tau}=x$ if and only if $Ax=x$. It is easy to see that

$$\{x\in \boldsymbol{Z}^9 \mid Ax=x\}=\boldsymbol{Z}\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}\oplus\boldsymbol{Z}\begin{pmatrix}0\\1\\0\\0\\-1\\1\\1\\-2\\-1\end{pmatrix}\oplus\boldsymbol{Z}\begin{pmatrix}0\\0\\1\\1\\-1\\-2\\0\\1\\-1\end{pmatrix}.$$

So if we put $\alpha=\theta_p-\theta_p\theta_q+\theta_p^{\sigma}\theta_q+\theta_q^{\tau}-2\theta_p\theta_q^{\tau}-\theta_p^{\sigma}\theta_q^{\tau}$, then we have $\alpha^{\sigma\tau}=\alpha$.

Let θ be a root of $f_1(X)$ or $f_2(X)$. We can test whether $\alpha\in\boldsymbol{Q}(\theta)$ as follows. We see that $\theta^{\sigma}\neq\theta$ because $\theta\notin k_q$. Hence $\{1,\theta,\theta^{\sigma}\}$ forms an integral basis of $\boldsymbol{Q}(\theta)$ over $\boldsymbol{Z}$. If α is contained in $\boldsymbol{Q}(\theta)$, there exist integers x_i such that $x_0+x_1\theta+x_2\theta^{\sigma}=\alpha$ and we can obtain x_i by solving a linear equation

$$(1)\qquad \begin{pmatrix}1&\theta&\theta^{\sigma}\\ 1&\theta^{\sigma}&\theta^{\sigma^2}\\ 1&\theta^{\sigma^2}&\theta\end{pmatrix}\begin{pmatrix}x_0\\x_1\\x_2\end{pmatrix}=\begin{pmatrix}\alpha\\ \alpha^{\sigma}\\ \alpha^{\sigma^2}\end{pmatrix}$$

approximately. Hence, if the solutions of (1) are not integers, then $\alpha\notin\boldsymbol{Q}(\theta)$.

If we let θ be a root of $f_1(X)$ and define θ^{σ} to be $71/3-(3/5)\theta-(1/15)\theta^2$, then we obtain x_i which are close to integers and get $\alpha=\theta$ by rounding x_i to integers. In other cases, x_i are not integers. Hence we can conclude that $<\sigma\tau>$ corresponds to $f_1(X)$ and $<\sigma\tau^2>$ corresponds to $f_2(X)$.

It is easy to calculate the unit group $\boldsymbol{E}_k$ and it is a routine work to test whether a rational number is contained in $\boldsymbol{E}_k k^{\times 3}$. Let k_i be the splitting field of $f_i(X)$. We have $3\cdot 223^2\notin \boldsymbol{E}_{k_1}k_1^{\times 3}$ and $3\cdot 223\in \boldsymbol{E}_{k_2}k_2^{\times 3}$. Therefore $\lambda_3(k_1)=0$. We do not know whether $\lambda_3(k_2)$ is zero.

5. Corrigendum of [2]. In the previous paper [2], we proved Corollary 2.3 using Lemma 3.2. But, after publication, we found that our proof for Lemma 3.2 includes some gaps. So we give here another proof of Corollary 2.3 without using Lemma 3.2. We use the same notations as in [2]. There exists the unique subfield k of k_pk_q in which ℓ splits. It is enough to show that $\lambda_{\ell}(k)=\mu_{\ell}(k)=0$. In the case $p\not\equiv 1 \pmod{\ell^2}$, $\mathfrak{p}_p$ and $\mathfrak{p}_q$ inert in the cyclotomic $\boldsymbol{Z}_{\ell}$-extension of k and become principal in $k\boldsymbol{Q}_1$. So we have $\lambda_{\ell}(k)=\mu_{\ell}(k)=0$ from Corollary 3.6 of [3]. On the other hand, we can handle the case $p\equiv 1 \pmod{\ell^2}$ by Theorem 1 of [4].

6. Tables. In the case $\ell=3$, we checked the conditions of Theorems 2.1 and 2.2 for all p and q such that $pq<100000$. We summarize our results as Tables I and II. There are two cyclic cubic fields k in which two prime numbers p and q are ramified. Such k is given as the splitting field of

$$X^3-X^2+\frac{1-pq}{3}X-\frac{1-3pq+pqu}{27},$$

where u and v are integers satisfying $4pq=u^2+27v^2$, $u\equiv 2 \pmod 3$, $u\equiv v \pmod 2$ and $v>0$. When $(\frac{\ell}{q})_{\ell}\neq 1$, one of these corresponds to $<\sigma\tau>$ and the other corresponds to $<\sigma\tau^2>$. For such k, we have $p^aq^b\in \boldsymbol{E}_k k^{\times 3}$ for some pair of integers a and b which satisfies $a\not\equiv 0 \pmod 3$ or $b\not\equiv 0 \pmod 3$. We notice that $p^aq^b\in\boldsymbol{E}_k k^{\times 3}$ implies $p^cq^d\notin \boldsymbol{E}_k k^{\times 3}$ for all pairs (c,d) such that $(c,d)\not\equiv(a,b),(2a,2b) \pmod 3$ because $B(k)$ is a cyclic group of order 3 generated by $\mathrm{cl}(\mathfrak{p}_p)$ and $\mathrm{cl}(\mathfrak{p}_q)$. In Table II, z is the element of $\boldsymbol{F}_3$ such that $pq^z\equiv 1 \pmod 9$. The asterisks in Tables I and II mean that we can apply none of Theorems 2.1 and 2.2.

Table I. The case $(\frac{\ell}{q})_\ell = 1$

p	q	u	v	a	b	$\lambda_3(k)$	u	v	a	b	$\lambda_3(k)$
13	103	−73	1	1	1	0	8	14	1	1	0
7	853	−139	13	0	1	0	104	22	0	1	0
13	499	−1	31	1	1	0	161	1	1	1	0
7	1021	−169	1	1	1	0	155	13	1	1	0
13	619	−79	31	0	1	0	164	14	0	1	0
43	193	11	35	0	1	0	173	11	0	1	0
13	853	−34	40	1	0	•	209	5	1	0	•
7	2029	−160	34	0	1	0	83	43	0	1	0
241	61	−211	23	0	1	0	194	28	0	1	0
277	61	−175	37	1	0	•	149	41	1	0	•
7	2617	−76	50	1	0	•	167	41	1	0	•
283	67	−1	53	1	1	0	161	43	1	1	0
313	61	−58	52	1	2	0	23	53	1	2	0
31	619	−277	1	1	1	0	209	35	1	1	0
7	3067	−274	20	1	1	0	293	1	1	1	0
7	3109	−295	1	1	1	0	272	22	1	2	0
7	3319	−265	29	1	1	0	302	8	1	1	0
349	67	−133	53	1	0	•	−52	58	1	0	•
7	3373	−211	43	1	2	0	113	55	1	2	0
43	643	−322	16	1	1	0	2	64	1	1	0
13	2131	−268	38	1	0	•	−187	53	1	0	•
211	151	−79	67	0	1	0	326	28	0	1	0
31	1093	−280	46	1	2	0	368	2	1	2	0
7	4957	−328	34	1	2	0	239	55	1	2	0
13	2803	116	70	1	2	0	359	25	1	2	0
7	5839	−337	43	0	1	0	230	64	1	0	•
409	103	−1	79	1	1	0	404	14	1	1	0
283	151	−385	29	1	1	0	−223	67	1	1	0
7	6271	−391	29	1	1	0	419	1	1	1	0
7	6637	−223	71	1	0	•	344	50	1	0	•
13	3739	161	79	0	1	0	404	34	0	1	0
7	7027	−202	76	1	0	•	41	85	1	0	•
331	151	−274	68	1	0	•	293	65	1	0	•
13	4057	−259	73	1	0	•	389	47	1	0	•
877	61	−70	88	1	2	0	11	89	1	2	0
31	1759	−343	61	1	1	0	467	1	1	1	0
907	61	−436	34	1	0	•	455	23	1	0	•
7	7951	−421	41	1	1	0	470	8	1	2	0
7	8233	−211	83	1	2	0	356	62	1	2	0
7	8527	−358	64	1	1	0	209	85	1	1	0
409	151	92	94	1	1	0	497	1	1	1	0
7	9421	−400	62	0	1	0	491	29	0	1	0
31	2131	101	97	0	1	0	263	85	0	1	0
7	9619	−202	92	0	1	0	365	71	0	1	0
157	439	−520	14	1	2	0	128	98	1	2	0
31	2383	−430	64	1	1	0	542	8	1	1	0
31	2389	−523	29	1	0	•	287	89	1	0	•
211	367	−430	68	1	0	•	−25	107	1	0	•
409	193	−559	11	0	1	0	−154	104	0	1	0
13	6079	−415	73	1	2	0	395	77	1	2	0
31	2713	125	109	1	0	•	287	97	1	0	•
823	103	−478	64	1	1	0	413	79	1	1	0
139	619	74	112	1	1	0	317	95	1	1	0
7	12391	−589	1	1	1	0	545	43	1	1	0
43	2029	65	113	0	1	0	551	41	0	1	0
7	13063	−601	13	1	2	0	533	55	1	2	0
607	151	425	83	1	1	0	506	64	1	1	0
139	661	−604	10	1	0	•	−523	59	1	0	•
31	3067	−529	61	1	1	0	605	23	1	1	0
1429	67	−25	119	1	0	•	380	94	1	0	•
97	997	−67	119	0	1	0	257	109	1	1	0
7	13831	−526	64	0	1	0	365	97	1	0	•
223	439	230	112	0	1	0	554	56	0	1	0
1471	67	−271	109	1	0	•	−109	119	1	0	•
1627	61	−121	119	1	2	0	41	121	1	2	0

Table II. The case $(\frac{\ell}{q})_\ell \neq 1$

			$\langle \sigma\tau \rangle$					$\langle \sigma\tau^2 \rangle$				
p	q	z	u	v	a	b	$\lambda_3(k)$	u	v	a	b	$\lambda_3(k)$
7	223	2	−13	15	1	2	•	41	13	1	2	0
7	337	1	92	6	1	1	•	−97	1	1	1	0
7	421	2	104	6	1	1	0	−85	13	1	1	•
13	229	2	−1	21	1	1	0	−109	1	1	1	•
7	463	1	83	15	1	1	•	−106	8	1	1	0
7	673	2	113	15	1	0	0	−76	22	0	1	0
7	769	1	−43	27	0	1	0	92	22	0	1	0
13	421	1	47	27	1	1	•	−142	8	1	1	0
79	97	2	−148	18	1	1	0	−175	1	1	1	•
31	349	1	47	39	1	2	0	101	35	1	2	•
13	859	2	−181	21	1	1	0	116	34	1	0	0
31	373	2	−163	27	1	1	0	215	1	1	1	•
7	1723	1	−169	27	1	1	•	209	13	1	1	0
79	157	1	173	27	1	2	0	65	41	1	1	0
7	1777	1	209	15	1	0	0	−223	1	1	1	0
13	1201	2	242	12	1	0	0	53	47	1	0	0
7	2239	2	−22	48	1	0	0	113	43	1	0	0
7	2311	2	50	48	1	2	•	−139	41	1	2	0
43	409	1	11	51	1	1	•	−259	11	1	1	0
13	1483	1	83	51	1	1	•	−268	14	1	1	0
43	457	2	176	42	1	2	•	−256	22	1	2	0
7	3037	1	281	15	1	2	0	−232	34	1	2	•
7	3163	1	29	57	1	0	0	83	55	1	0	0
79	283	1	245	33	1	1	•	299	1	1	1	0
13	1741	2	53	57	1	1	0	−298	8	1	1	•
13	1789	1	−151	51	0	1	0	−259	31	0	1	0
79	337	1	137	57	1	2	0	−52	62	1	2	•
7	3823	2	−139	57	1	2	•	239	43	1	2	0

7	3877	2	−328	6	1	0	0	293	29	1	0	0
7	3919	1	218	48	0	1	0	−295	29	0	1	0
79	349	2	−301	27	1	0	0	−328	10	0	1	0
13	2137	2	−118	60	1	2	*	−307	25	1	2	0
7	4129	2	167	57	1	0	0	−265	41	1	0	0
97	313	2	−319	27	1	1	0	−346	8	1	1	*
7	4507	2	302	36	0	1	0	−211	55	0	1	0
79	409	1	−331	27	0	1	0	−196	58	0	1	0
7	4621	1	29	69	1	1	*	−160	62	1	1	0
97	337	1	47	69	1	1	*	−142	64	1	0	0
31	1069	1	−232	54	1	2	0	−16	70	1	2	*
7	4759	2	−337	27	1	1	0	365	1	1	1	*
13	2731	2	233	57	1	1	0	287	47	1	1	*
13	2887	1	−376	18	0	1	0	−79	73	0	1	0
139	277	1	353	33	1	2	0	245	59	1	1	0
7	5641	2	−265	57	1	0	0	356	34	1	0	0
13	3121	1	398	12	1	0	0	47	77	0	1	0
7	5881	1	20	78	1	1	*	−169	71	1	1	0
13	3229	1	−385	27	1	2	0	155	73	0	1	0
13	3271	2	287	57	1	2	*	−307	53	1	2	0
7	6133	1	407	15	1	2	0	−349	43	1	2	*
13	3307	2	404	18	0	1	0	−109	77	1	0	0
97	463	1	−124	78	1	2	0	−313	55	1	2	*
13	3541	2	−415	21	1	1	0	125	79	1	1	*
13	3613	2	−343	51	1	1	0	278	64	1	1	*
13	3697	1	434	12	0	1	0	−79	83	0	1	0
31	1579	2	314	60	1	1	0	−442	4	1	1	*
139	373	2	−55	87	1	0	0	−244	74	1	2	0
13	4003	1	443	21	1	1	*	92	86	1	1	0
157	337	2	413	39	1	1	0	−343	59	1	1	*
79	673	2	−220	78	1	2	*	−409	41	1	2	0
7	7699	1	−295	69	1	1	*	407	43	1	1	0
13	4243	2	44	90	1	2	*	−469	5	1	2	0
43	1291	1	−133	87	1	2	0	−403	47	1	2	*
97	601	2	−355	63	1	0	0	−139	89	1	0	0
157	373	2	287	75	1	2	*	−469	23	1	2	0
13	4597	1	389	57	1	2	0	281	77	1	2	*
31	1951	1	461	33	1	0	0	−457	35	1	0	0
79	769	1	−484	18	1	1	*	−349	67	1	1	0
223	277	2	−472	30	1	0	0	−256	82	1	0	0
13	4759	1	−421	51	1	2	0	281	79	1	2	*
31	2011	2	−469	33	1	0	0	233	85	1	0	0
223	283	1	−394	60	1	2	0	−502	4	1	2	*
229	283	2	−460	42	0	1	0	−244	86	0	1	0
79	823	1	−106	96	1	2	0	191	91	1	2	*
7	9463	1	−421	57	1	0	0	281	83	1	0	0
13	5101	1	−25	99	1	1	*	515	1	1	1	0
7	9547	2	482	36	1	1	0	−517	1	1	1	*
79	859	1	−259	87	1	2	0	38	100	1	2	*
7	9787	1	−97	99	1	2	0	281	85	1	2	*
79	877	1	515	21	1	1	*	488	38	1	1	0
7	10093	1	344	78	0	1	0	−169	97	0	1	0
13	5641	1	−538	12	1	2	0	83	103	1	2	*
7	10723	1	461	57	1	2	0	−538	20	1	2	*
13	5881	2	467	57	1	0	0	278	92	1	0	0
13	5953	2	−109	105	1	2	*	−541	25	1	2	0
7	11131	2	−526	36	0	1	0	419	71	0	1	0
13	6007	2	−541	27	1	1	0	161	103	1	1	*
7	11311	2	545	27	1	1	0	−454	64	1	1	*
13	6163	1	−151	105	0	1	0	551	25	0	1	0
13	6397	1	−187	105	1	1	*	−538	40	1	1	0
7	11887	2	−13	111	1	1	0	−202	104	1	1	*
31	2689	1	−556	30	1	0	0	308	94	1	0	0
139	601	1	191	105	1	1	*	407	79	1	1	0
7	11941	2	41	111	0	1	0	176	106	0	1	0
7	12517	2	293	99	0	1	0	−463	71	0	1	0
13	6781	2	−577	27	1	1	0	−118	112	1	1	*
283	313	1	533	51	1	0	0	587	19	1	0	0
7	12739	1	153	111	1	0	0	−358	92	1	0	0
7	13441	1	209	111	1	1	*	20	118	1	1	0
157	601	1	−250	108	1	2	0	614	4	1	0	0
7	13873	1	−412	90	1	1	*	209	113	1	1	0
79	1231	2	14	120	1	2	*	−337	101	1	1	*
7	13903	2	−22	120	1	1	0	−211	113	1	1	*
31	3163	2	278	108	0	1	0	170	116	0	1	0
43	2281	1	560	54	0	1	0	128	118	0	1	0
283	349	1	−619	21	1	1	*	−565	53	1	1	0

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