On smooth projective threefolds with non-trivial surjective endomorphisms

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The main purpose of this paper is to announce the structure of non-singular projective threefold X with a surjective morphism $f: X \to X$ onto itself, which is not an isomorphism.

We call it a non-trivial surjective endomorphism of X.

The details will be published elsewhere.

Lemma 1. Let $f: X \to X$ be a surjective morphism from a non-singular projective variety X onto itself. Then

(1) f is a finite morphism.

(2) If $\kappa(X) \geq 0$, f is a finite étale morphism.

(3) If f is a finite étale morphism, then $\chi(\mathcal{O}_X) = \deg(f) \cdot \chi(\mathcal{O}_X)$.

The structure of algebraic surfaces with a non-negative Kodaira dimension, which admit a non-trivial surjective endomorphism, are fairly simple. They are minimal and by taking s finite étale covering, isomorphic to an abelian surface or a direct product of an elliptic curve and a smooth curve of genus ≥ 2. In this note, we are mainly concerned with the case where X is a smooth projective threefold with non-negative Kodaira dimension $\kappa(X)$. Contrary to the case of algebraic surfaces, they are not necessarily minimal, but similar results also hold in this case. We cannot drop the assumption that $f: X \to X$ is a morphism. There are infinitely many examples which admit a generically finite rational map $f: X \dashrightarrow X$ of degree ≥ 2, eg. a Kummer surface, or a relatively minimal elliptic surface with a global section.

Notations. In the present note, by a smooth projective n-fold X, we mean a non-singular projective manifold of dimension n defined over $\boldsymbol{C}$.

K_X : the canonical bundle of X

$\kappa(X)$: the Kodaira dimension of X

$\chi(\mathcal{O}_X)$: the Euler-Poincaré characteristic of the structure sheaf $\mathcal{O}_X$

$N_1(X) := (\{1\text{-cycles on } X\})/_{\equiv} \otimes_{Z} \boldsymbol{R}$, where $\equiv$ means a numerical equivalence.

$NE(X) :=$ the smallest convex cone in $N_1(X)$ containing all effective 1 - cycles.

$\overline{NE}(X) :=$ Kleiman-Mori cone of X, ie. the closure of $NE(X)$ in $N_1(X)$ for the metric topology.

$\rho(X) := \dim_R N_1(X)$, the Picard number of X.

The next propositions, which are direct consequences of Mori theory [1], play a key role in this paper.

Proposition 2. Let $f: Y \to X$ be a finite, étale covering between smooth projective n-folds with $\rho(X) = \rho(Y)$.

Then $f^*: N_1(X) \to N_1(Y)$ (resp. $f_*: N_1(Y) \to N_1(X)$) is isomorphic and $f^*\overline{NE}(X) = \overline{NE}(Y)$ (resp. $f_*\overline{NE}(Y) = \overline{NE}(X)$).

Moreover, if the canonical bundle K_X of X (hence $K_Y \simeq f^*K_X$) is not nef, there is a one to one correspondence between

{extremal rays on $\overline{NE}(X)$} and {extremal rays on $\overline{NE}(Y)$} under the above isomorphisms f^* and f_*.

Proposition 3. Let X, Y be non-singular projective threefolds with non-negative Kodaira dimensions. Assume that X and Y have the same Picard number and there exists a finite étale covering $f: Y \to X$, which is not an isomorphism.

If the canonical bundle K_X of X (hence $K_Y \simeq f^*(K_X)$) is not nef, then Mori's extremal contractions of X (resp. Y), $\mathrm{Cont}_R: X \to X'$ (resp. $\mathrm{Cont}_{\tilde{R}}: Y \to Y'$), associated to each extremal ray R of $\overline{NE}(X)$ (resp. $\tilde{R}$ of $\overline{NE}(Y)$), is a birational divisorial contraction, which is (inverse of) the blow-up along a smooth curve C (resp. $\tilde{C}$) on X' (resp. Y'). Moreover C (resp. $\tilde{C}$) is not $\boldsymbol{P}^1$ and if $f^*(R) = \tilde{R}$, the following commutative diagram holds.

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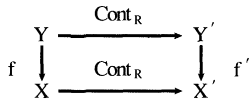
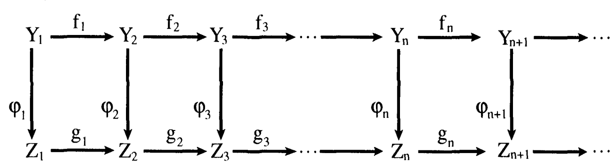


where $f' : Y' \to X'$ is a non-isomorphic finite étale covering, and $\tilde{C} = f'^{-1}(C)$.

Proposition 4. Let $Y_1 \overset{f_1}{\to} Y_2 \overset{f_2}{\to} \cdots \to Y_n \overset{f_n}{\to} Y_{n+1} \overset{f_{n+1}}{\to} \cdots$ be an infinite descending sequence of non-isomorphic,finite étale coverings between non-singular projective threefolds Y_n with non-negative Kodaira dimensions. $(n = 1, 2, \cdots)$
Moreover we assume that the canonical bundle K_{Y_n} of Y_n is not nef and the Picard number $\rho(Y_n)$ of Y_n $(n = 1, 2, \cdots)$ are constant. Then Mori's extremal contraction of Y_n, $\varphi_n := \mathrm{Cont}_{R_n} : Y_n \to Z_n$, associated to each extremal ray R_n of $\overline{NE}(Y_n)$, is the birational divisorial contraction, which is (inverse of) the blow-up along a smooth *elliptic curve* E_n on Z_n. Moreover, if $R_n = (f_n \circ \cdots \circ f_1)_* R_1$ for all $n = 1, 2, \cdots$, the following commutative diagram holds.



where $g_n : Z_n \to Z_{n+1}$ is a non-isomorphic, finite, étale covering and $E_n = g_n^{-1}(E_{n+1})$ holds for all n.

Remark 5. For a non-trivial surjective endomorphism $f : X \to X$ between a non-singular projective threefold X with $\kappa(X) \geq 0$ and whose K_X is not nef, all the assumptions in proposition 4 are automatically satisfied by putting $Y_n := X$ and $f_n := f$ for all n.

Proposition 6 (Minimal reduction). Let X be a non singular projective threefold with a non-negative Kodaira dimension. Assume that there exists a non-isomorphic surjective endomorphism $f : X \to X$ and K_X is not nef. Then after a finite number of extremal divisorial contractions as in Proposition 4, we obtain $f_n : Y_n \to X_n$, which is birational to $f : X \to X$. Y_n and X_n are non-singular minimal models of X and f_n is a finite étale covering, which is not an isomorphism. (Hence they are isomorphic in codimension one and connected by a sequence of flops by Kawamata [3] and Kollar [5].)

By the abundance theorem by Miyaoka [4] and Kawamata [2], K_{X_n} and K_{Y_n} are semi-ample. If $\kappa(X) = 0$ and 2, thanks to the Bogomolov's decomposition theorem [6] and the standard fibration theorem by Nakayama [7,8], X_n and Y_n are isomorphic and we can describe the structure of the Iitaka fibration of them completely.

The following proposition is well-known.

Proposition 7. Let $f : X \to X$ be a surjective morphism between a non-singular projective n-fold of general type.
Then f is an isomorphism.

Combining Propositions 4,6,7 and the above remark, we obtain our Main Theorem.

Main Thorem (A). Let X be a non-singular projective threefold with a non-negative Kodaira dimension, which admits a non-isomorphic, surjective endomorphism $f : X \to X$.

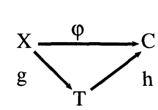
Then the minimal models of X are non-singular, unique up to isomorphisms and one of the following cases occurs.

Case 1) If $\kappa(X) = 2$, X has the structure of an elliptic fiber space $\varphi : X \to T$ over a normal surface T with at most quotient singularities and f induces an automorphism of the base space T and is compatible with φ. Moreover,

1a) X has the structure of a Seifert fiber space over T. (i.e. φ is equi-dimensional and X has at most multiple singular fibers of type ${}_mI_0$ in the sense of Kodaira, and is a principal fiber bundle outside them. K_X is numerially φ-trivial.)

1b) By taking a suitable finite étale covering $\tilde{X}$ of X, $\tilde{X}$ is isomorphic to the direct product $\tilde{T} \times E$, where $\tilde{T}$ is a smooth algebraic surface of general type (not necessarily minimal) and E is a smooth elliptic curve.

Case 2) If $\kappa(X) = 1$, the minimal model X' of X is non-singular, and the general fiber of the Iitaka fibration $\Phi_{|mK_{X'}|} : X' \to C$ is isomorphic to an Abelian surface or a hyperelliptic surface. In the latter case, we obtain the following commutative diagram of fiber spaces,



where $g: X \to T$ is a Seifert elliptic fiber space over a normal surface T with at most quotient singularities, $h: T \to C$ is a $\boldsymbol{P}^1$ or elliptic fiber space and the general fiber of $\varphi := h \circ g$ is a hyperelliptic surface. f induces an endomorphism (resp. automorphism) of T (resp. C), and is compatible with φ, g and h. By taking a finite étale covering $\tilde{X}$ of X, $\tilde{X}$ is isomorphic to the direct product $\tilde{T} \times E$, where $\tilde{T}$ is a smooth algebraic surface with $\kappa(\tilde{T}) = 1$ and E is an elliptic curve.

Case 3) If $\kappa(X) = 0$, by taking a suitable finite étale covering $\tilde{X}$ of X, $\tilde{X}$ is isomorphic to an Abelian threefold or the direct product $\tilde{T} \times E$, where $\tilde{T}$ is a smooth algebraic surface which is birational to an abelian surface or a $K3$ surface, and E is an elliptic curve. In the latter case, X has the structure of a Seifert elliptic fiber space over the quotient of $\tilde{T}$ by a finite group G.

f is compatible with it and induces an automorphism of the base space $\tilde{T}/G$.

In cases (1) and (3), $f: X \to X$ can be lifted to an endomorphism $\tilde{f}: \tilde{X} \to \tilde{X}$ and if moreover, $\tilde{X}$ is of split type, it is compatible with the first projection $\tilde{X} := \tilde{T} \times E \to \tilde{T}$ and induces an automorphism of $\tilde{T}$.

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