Class number two problem for real quadratic fields with fundamental units with the positive norm

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1. Introduction and notations. Throughout this paper, we denote by N the set of positive rational integers, and put $N_0 = N \cup \{0\}$. Z will mean as usual the set of rational integers. For a square-free $D \in N$, the real quadratic field $Q(\sqrt{D})$ will be denoted by k, its class number by h_k and its fundamental unit > 1 by $\varepsilon_D = (t + u\sqrt{D})/2$. The norm map from k to Q will be denoted by N. The class number two problem requires to determine the set of all D for which $h_k = 2$ under certain conditions. This problem was solved by *Katayama* [2,3] with one possible exception for the conditions $N\varepsilon_D = -1$, $1 \leq u \leq 200$; by *Mollin and Williams* [5] for k of *Extended Richaud-Degert* type (i.e. with $D = m^2 + r$ where $4m \equiv 0 \pmod{r}$), also with one possible exception; and by *Taya and Terai* [7] for k of *Narrow Richaud-Degert* type (i.e. with $r = \pm 1$ or ± 4).

In this paper, we shall consider this problem for the case $N\varepsilon_D = 1$, $1 \leq u \leq 100$ and solve it with one possible exception (see Theorem below).

2. Lemmas and propositions. We begin by citing two known results as Lemmas 1,2 (The letters N, D, ε_D, t, u will always keep the meanings explained above. For a real number x, $[x]$ means as usual the greatest integer $\leq x$).

Lemma 1 (*Yokoi* [11]). Suppose $N\varepsilon_D = 1$. Then the following conditions for $n, v \in N_0$, $w \in Z$ determine these numbers uniquely, and we have $n = [t/u^2]$, $w = D - 2tn + u^2n^2$:

$$t = u^2n + v,\ v^2 - 4 = wu^2,\ v < u^2$$
$$D = u^2n^2 + 2vn + w.$$

For our real quadratic field $k = Q(\sqrt{D})$, we denote by d_k its discriminant (i.e. d_k is D or $4D$ according as $D \equiv 1 \pmod 4$ or $\equiv 2, 3 \pmod 4$), by χ_k Kronecker character of k and by $L(1, \chi_k)$ the Dirichlet L-function with this character.

Lemma 2 (*Tatuzawa* [6]). Suppose $d_k \geq \max(e^{1/\alpha}, e^{11.2})$ for a real number α with $0 < \alpha < 1/2$. Then we have

$$L(1, \chi_k) > \frac{0.655\alpha}{d_k^{\alpha}}$$

with one possible exception of k.

The following lemma will be used immediately afterward:

Lemma 3. We have $\varepsilon_D < 2u\sqrt{D}$.

Proof. This follows easily from $t = \sqrt{Du^2 \pm 4} < u\sqrt{D} + 2$. Q. E. D.

Let D be a square-free number $\in N$ for which $N\varepsilon_D = 1$ and n, v, w be the numbers $\in Z$ determined by the conditions in Lemma 1. From Lemmas 2,3, we can deduce the following

Proposition 1. D, n, v, w being as above, there exists a real number $\upsilon(u)$ determined by u, such that $h_k > 2$ follows from $n \geq \upsilon(u)$, with one possible exception of D.

Proof. From Lemma 2 and the well-known Dirichlet's class number formula, we get

$$h_k = \frac{\sqrt{d_k}}{2\log\varepsilon_D} L(1, \chi_k) > \frac{0.655}{2\log\varepsilon_D} \frac{\sqrt{d_k}\, d_k^{-1/y}}{y}$$

for $y \geq 11.2$ and $d_k \geq e^y$, with one possible exception of k. Since $\varepsilon_D < 2u\sqrt{D} \leq 2u\sqrt{d_k}$ by Lemma 3, we have

$$h_k > \frac{0.655 d_k^{1/2-1/y}}{y(\log d_k + 2\log u + 2\log 2)}.$$

y being fixed, the right-hand side is a monotone increasing function of d_k. Thus we can replace here d_k by e^y to obtain

$$h_k > \frac{0.655 d_k^{y/2-1}}{y(y + 2\log u + 2\log 2)}$$

Let us denote by $f_u(y)$ the right-hand side of this inequality. For any fixed u, $f_u(y)$ tends to ∞ as $y \to \infty$. So there exists a real number $c(u) \geq 11.2$ satisfying $f_u(c(u)) \geq 2$. Thus, solving the inequality

$$e^{c(u)} \leq D = u^2n^2 + 2vn + w \leq d_k$$

for n, one can find a real number $\upsilon(u)$ such that $h_k > f_u(c(u)) \geq 2$ for $n \geq \upsilon(u)$. Q. E. D.

In fact, we may take $\upsilon(u) \geq \sqrt{4 + u^2e^{c(u)}}/u^2$. Moreover, we can choose $c(u) < 16.5$ for 1

$\leq u \leq 100$ by the help of computer, so that we obtain

$$\sqrt{4 + u^2 e^{c(u)}} < \sqrt{4 + u^2 e^{16.5}} < 3828u$$

and can put $\upsilon(u) = 3828/u$ for such u's. This result will be soon used.

To facilitate the formulation of the next Lemmas 4,5, we introduce the following

Definition. For many $m \in \mathrm{N}$ and square-free $D \in \mathrm{N}$, the Diophantine equation $x^2 - Dy^2 = \pm 4m$ is said to have a trivial solution (x_0, y_0) if $m = s^2$ and s divides both x_0 and y_0. Any other solution is called non-trivial.

Lemma 4 (*Davenport- Ankeny- Hasse- Ichimura*). The notations being as above from the existence of at least one non-trivial solution of $x^2 - Dy^2 = \pm 4m$ follows $m \geq (t-2)/u^2$.

Proof. See [10] Lemma 1. Q. E. D.

Lemma 5. Let D, k be as above, q an odd prime with $\left(\frac{D}{q}\right) = 1$ and e the order of the ideal class of k containing a prime factor of q. Then the Diophantine equation $x^2 - Dy^2 = \pm 4q^e$ has a non-trivial solution.

Proof. Let Q be a prime factor of q in k and put $Q^e = (w)$, $w = (x + y\sqrt{D})/2$. Since q splits in k, we get

$$q^e = NQ^e = |N(w)| = \frac{|x^2 - 4y^2|}{4}. \quad \text{Q. E. D.}$$

Proposition 2. Let D be as above, n, v, w the numbers given in Lemma 1, and q an odd prime with $\left(\frac{D}{q}\right) = 1$. If $h_k = 2$, then $q^2 \geq n$.

Proof. By Lemmas 4,5, we have $q^e \geq (t-2)/u^2$. Here we may replace e by 2 owing to $h_k = 2$. Therefore by Lemma 1, we get

$$q^2 \geq \frac{u^2 n + v - 2}{u^2} = n + \frac{v-2}{u^2} \geq n - \frac{2}{u^2}.$$

If $u \geq 2$, we have $q^2 \geq n - 1/2$ whence $q^2 \geq n$. If $u = 1$, we have $q^2 \geq n - 2$ and $D = n^2 - 4$

Table

(u,	D)	(u,	D)	(u,	D)	(u,	D)	(u,	D)
(1,	165)*	(7,	429)*	(13,	4245)*	(24,	8357)	(56,	111)
(1,	221)*	(7,	1205)*	(14,	51)*	(27,	6573)*	(56,	305)
(1,	285)*	(7,	1245)*	(14,	447)*	(28,	194)*	(56,	602)
(1,	357)*	(7,	2373)*	(15,	2013)*	(30,	1007)	(56,	782)*
(1,	957)*	(7,	5885)*	(15,	2037)*	(32,	258)*	(56,	5397)
(1,	1085)*	(7,	8333)*	(15,	5117)	(32,	1605)*	(57,	1005)
(1,	1517)*	(8,	39)*	(15,	5645)*	(32,	7733)*	(57,	6773)
(1,	2397)*	(8,	95)*	(16,	66)*	(33,	3893)	(58,	843)*
(2,	15)*	(8,	105)*	(16,	395)*	(34,	287)*	(60,	70)
(2,	35)*	(8,	138)*	(16,	2717)*	(35,	861)	(60,	902)*
(2,	143)*	(8,	203)*	(16,	5757)*	(35,	1653)	(64,	1022)*
(3,	205)*	(8,	885)	(17,	2613)*	(40,	155)	(64,	2301)*
(3,	1469)*	(8,	1173)*	(19,	3237)*	(40,	402)*	(65,	11357)
(3,	1965)*	(8,	2093)*	(19,	9005)*	(40,	2261)	(66,	335)
(3,	2085)*	(8,	3813)*	(20,	102)*	(40,	4893)*	(69,	2877)
(3,	2669)	(9,	741)*	(20,	222)	(42,	923)	(72,	183)
(4,	30)*	(9,	2045)*	(21,	1581)	(44,	482)*	(72,	1298)*
(4,	42)*	(10,	635)*	(22,	119)*	(45,	5453)	(80,	3597)*
(4,	110)*	(11,	3005)*	(22,	123)*	(46,	527)*	(84,	266)
(4,	182)*	(11,	5957)*	(23,	4773)*	(48,	299)	(88,	273)
(5,	645)*	(12,	34)*	(24,	55)	(48,	3605)*	(88,	755)
(5,	4277)*	(12,	78)	(24,	146)*	(48,	7973)	(95,	1749)
(5,	7157)*	(12,	230)*	(24,	327)*	(50,	623)*	(96,	710)
(6,	87)*	(12,	318)*	(24,	377)	(51,	805)	(96,	14405)*
(6,	215)*	(13,	1533)*	(24,	2765)	(52,	678)*	(99,	1837)

by Lemma 1. If $q^2 = n - 1$ or $n - 2$, $D = n^2 - 4$ should be divisible by 4 or q^2 respectively in contradiction to choice of D. Therefore $q^2 \geq n$.

Q. E. D.

Suppose now $D \in \mathrm{N}$ is square-free and $N\varepsilon_D = 1$. Let n, v, w be the numbers given in Lemma 1 and $1 \leq u \leq 100$. From Proposition 1,2 and the genus theory follow the following necessary conditions for $h_k = 2$:

(i) $0 \leq n < \upsilon(u) = 3828/u$,

(ii) $q^2 \geq n$ for the least odd prime q with $\left(\frac{D}{q}\right) = 1$,

(iii) The number of distinct prime factors of d_k is 2 or 3.

3. Main theorem. We have now all necessary tools to get the following

Theorem. There exists exactly 125 real quadratic fields $k = Q(\sqrt{D})$ as given in the Table (with one possible exception) with class number 2 with $1 \leq u \leq 100$, where $(t + u\sqrt{D})/2$ is the fundamental unit > 1 of k.

Proof. By the help of a computer and using *Kida's UBASIC 86*, we can list up all D satisfying the above necessary conditions with $h_k = 2$.

Remark. In the Table, those given in [5] are marked with *.

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