The semilattices of nilextensions of left groups and their varieties

By Zhonghao JIANG and Liming CHEN

Department of Mathematics, Zhanjiang Teachers' College, Zhanjiang
Guangdong 524048, People's Republic of China

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Abstract: In this paper, we characterize the semigroups which are the semilattices of nilextensions of left groups and prove that these semigroups form a variety defined by the equation $(xy)^{\omega}(yx)^{\omega} = (xy)^{\omega}$. Moreover, we obtain some decompositions of this variety by Mal'cev product and semidirect product. In particular, we prove that this variety is just the semidirect product $\mathbf{G}*\mathbf{R}$ of the variety $\mathbf{G}$ of groups and the variety $\mathbf{R}$ of $\mathcal{R}$-trivial semigroups.

1. Introduction and preliminaries. A semigroup S is called a semilattice Y of semigroups of type T if there is a homomorphism Ψ from S onto a semilattice Y and the inverse image of each element of Y under Ψ is a semigroup of type T. In [3], Davenport studied the semigroups which are semilattices of nilextensions of groups. The purpose of this paper is to characterize the class of finite semigroups that are semilattices of nilextensions of left groups and to generalize some results of [3]. For simplicity's sake, we denote by **SNLG** the class of finite semigroups which are semilattices of nilextensions of left groups.

First, we will characterize the semigroups in **SNLG** and prove that **SNLG** is defined by equation $(xy)^{\omega}(yx)^{\omega} = (xy)^{\omega}$. Hence **SNLG** is a variety in the sense of Eilenberg [4] since it is closed under finite products, subsemigroups and homomorphic images. Then, using the functorially minimal L' homomorphic image, a concept introduced in [1], we give a decomposition of **SNLG** by semidirect product $\mathbf{G}*\mathbf{R}$. We also consider some decompositions of **SNLG** by Mal'cev products.

For convenience let us review some definitions and facts germane to what follows. If S is a semigroup, then S^1 is the semigroup obtained by adjoining an identity element 1 to S. A semigroup S is said to be archimedean, if for any $a, b \in S$, there exist $m, n \in N$ such that $a^m \in S^1bS^1$, $b^n \in S^1aS^1$.

Lemma 1.1[8]. A semigroup S is a semilattice of archimedean semigroups if and only if, for every $a, b \in S$, the condition $b \in S^1aS^1$ implies $b^i \in S^1a^2S^1$ for some positive integer i.

A finite semigroup is a left group, if for any $x \in S$, $S = Sx$.

Lemma 1.2 [Theorem 1.27,2]. Let S be a semigroup. Then the following conditions are equivalent:

(1) S is a left group;

(2) S is a regular semigroup whose idempotent elements form a left zero semigroup;

(3) S is isomorphic to a direct product of a left zero semigroup and a group.

Let I be an ideal of S. S is a nilextension of I if for any $a \in S$, there exists a positive integer n such that $a^n \in I$.

Throughout this paper, all semigroups are finite.

For any element x in a semigroup S, we denote by x^{ω} the unique idempotent element in the subsemigroup of S generated by x.

A semigroup S is $\mathcal{R}$-trivial, if S satisfies the equation $(xy)^{\omega}x = (xy)^{\omega}$.

For a semigroup S, we denote by $E(S)$ the set of idempotents of S.

For non defined notations and terminology, the reader is referred to [5], [7], [9].

2. The main structure theorem.

Theorem 2.1. Let S be a semigroup. Then the following are equivalent:

(1) $S \in \mathbf{SNLG}$;

(2) Each regular $\mathcal{D}$-class of S is a left group;

(3) For all x and y in S, if xy and yx are idempotent, then $(xy)(yx) = xy$;

(4) S satisfies the equation $(xy)^{\omega}(yx)^{\omega} =$

$(xy)^{\omega}$.

Proof. (1) $\Rightarrow$ (2) Suppose that S is a semilattice of semigroups S_α, $\alpha \in Y$ with S_α being a nilextension of left group T_α. It is easy to show that if $x \mathscr{D} y$, then x and y are in the same S_α. But each S_α contains only one regular $\mathscr{D}$-class T_α. Hence each regular $\mathscr{D}$-class of S is a left group.

(2) $\Rightarrow$ (3) If xy and yx are idempotent, then $xy \mathscr{D} yx$. Hence xy and yx are in the same regular $\mathscr{D}$-class of S. Thus xy, yx are idempotent elements in a left group. By Lemma 1.2, $(xy)(yx) = xy$.

(3) $\Rightarrow$ (4) For any $x, y \in S$, $(xy)^{\omega} = x[(yx)^{\omega-1}y]$, $(yx)^{\omega} = [(yx)^{\omega-1}y]\,x$. By (3), $(xy)^{\omega}(yx)^{\omega} = (xy)^{\omega}$.

(4) $\Rightarrow$ (1) Let $a, b \in S$ with $b = uav$, $u, v \in S^1$. Then

$$b^{\omega} = (uav)^{2\omega} = ua(vua)^{\omega}v(uav)^{\omega-1}.$$

Since $(vua)^{\omega} = (vua)^{\omega}(avu)^{\omega}$, $b^{\omega} = ua(vua)^{\omega}\cdot(avu)^{\omega}v(uav)^{\omega-1} \in S^1a^2S^1$. By Lemma 1.1, S is a semilattice of archimedean semigroups S_α, $\alpha \in Y$. Since S_α is finite, S_α has the least ideal, and if one denotes it by T_α, then T_α is a finite simple semigroup. Hence T_α is completely simple and so T_α is a regular semigroup. Since S_α is archimedean, $E(S_\alpha) \subseteq T_\alpha$. Thus S_α is a nilextension of T_α. For any $x, y \in T_\alpha$, there exist $u, v, w \in T_\alpha$ such that $x = xux$, $ux = vyw$, as T_α is regular simple. Hence

$ux = (ux)^{\omega} = (vyw)^{\omega} = (vyw)^{\omega}(wvy)^{\omega} \in T_\alpha y$. Thus $x = xux \in T_\alpha y$. It follows that T_α is a left group and so S is a semilattice of nilextensions of left groups.

The following proposition, generalizing Proposition 1.6 of [3], gives us some information about **SNLG** that will be useful later.

Proposition 2.2. Let S be a semilattice of semigroups S_α, $\alpha \in Y$, with S_α being a nilextension of left group T_α for each $\alpha \in Y$. Define the relation ρ on S as follows:

$s\,\rho\,t$ iff there exist $\alpha \in Y$, $e \in E(S_\alpha)$ such that $s, t \in S_\alpha$ and $es = et$. Then ρ is a congruence on S and S/ρ is a Clifford semigroup.

Proof. It is easy to prove that ρ is an equivalence on S. Let $s\,\rho\,t$ and c be in S_β. Then there exist $\alpha \in Y$, $e \in E(S_\alpha)$ such that $s, t \in S_\alpha$ and $es = et$. Since $(ce)(sc) = (ce)(tc)$, $(ce)^{\omega}(sc) = (ce)^{\omega}(tc)$. Thus $sc\,\rho\,tc$.

Since $(ec)^{\omega}e\cdot(ec)^{\omega}e = (ec)^{\omega}e$ and $E(S_{\alpha\beta})$ is a left zero semigroup,

$$(ec)^{\omega}e = (ec)^{\omega}(ec)^{\omega}e = (ec)^{\omega}.$$

Therefore

$(ce)^{\omega}cs = c(ec)^{\omega}s = c(ec)^{\omega}es = c(ec)^{\omega}et = c(ec)^{\omega}t = (ce)^{\omega}ct$. Thus $cs\,\rho\,ct$. It follows that ρ is a congruence on S.

Next, we will show that S/ρ is a completely regular semigroup whose idempotents are in the center of S. In fact, for any $x \in S$, $x^{\omega}x = x^{\omega}\cdot x^{\omega+1}$. Hence $x\,\rho\,x^{\omega+1}$. It follows that S/ρ is a completely regular semigroup. For any $e \in E(S)$, by Theorem 2.1,

$(xe)^{\omega} = (xe)^{\omega}(ex)^{\omega} = (xe)^{\omega}e(xe)^{\omega-1}x = (xe)^{\omega}(xe)^{\omega-1}x = (xe)^{2\omega-1}x$.

Hence $(xe)^{\omega}\cdot xe = xe(xe)^{\omega} = (xe)\cdot(xe)^{2\omega-1}ex = (xe)^{2\omega}ex = (xe)^{\omega}ex$. It follows that $xe\,\rho\,ex$. Thus S/ρ is a Clifford semigroup.

3. Some decompositions of SNLG. First, we recall some definitions and facts which are also found in [7], [9]. A relational morphism $\tau : S \to T$ is a function from S to the power set of T with the property that for all s and t in S, $s\tau$ is not empty and $(s\tau)(t\tau) \subseteq (st)\tau$.

Let $\mathbf{U}$ and $\mathbf{V}$ be varieties of semigroups. Define the Mal'cev product of $\mathbf{U}$ and $\mathbf{V}$ to be the following class of semigroups:

$\mathbf{U}^{-1}\mathbf{V} = \{S$: there exists a surjective relational morphism τ from S to a semigroup T in $\mathbf{V}$ such that for each $e \in E(T)$, $e\tau^{-1} \in \mathbf{U}\}$.

Lemma 3.1 [7]. $\mathbf{U}^{-1}\mathbf{V}$ is a variety generated by the following class of semigroups:

$\{S$: there exists a surjective morphism Ψ from S to a semigroup T in $\mathbf{V}$ such that for each $e \in E(T)$, $e\Psi^{-1} \in \mathbf{U}\}$.

Let R and T be semigroups. A left action of T on R is a function: $T \times R \to R$, $(t, r) \to {}^{t}r$ satisfying the following conditions:

(1) ${}^{t}(rr') = {}^{t}r\cdot{}^{t}r'$;

(2) ${}^{t}({}^{t'}r) = {}^{tt'}r$;

for all $t, t' \in T$, $r, r' \in R$.

The semidirect product $R*T$ is the set $R \times T$ equipped with the product

$$(r, t)(r', t') = (r\,{}^{t}r', tt').$$

Let $\mathbf{U}$ and $\mathbf{V}$ be varieties of semigroups. Then $\mathbf{U} * \mathbf{V}$ is the variety generated by the following class of semigroups:

$$\{R*T : R \in \mathbf{U},\ T \in \mathbf{V}\}$$

Let J be a $\mathscr{J}$-class of a semigroup S. Let $F(J) = \cup\ \{J' : J'$ is a $\mathscr{J}$-class of S and $J \nleq J'\}$. Then $F(J)$ is an ideal of S (If we also assume that the empty set is an ideal of S). Denote by π the canonical homomorphism from S onto $S/F(J)$. Consider the relation $\equiv_R$ on $S/F(J)$ defined by

$$s_1^{\pi} \equiv_R s_2^{\pi} \text{ iff } (x^{\pi})(s_1^{\pi}) \mathscr{L} (x^{\pi})(s_2^{\pi})$$

in $S/F(J)$ for all $x \in J$. Then $\equiv_R$ is a congruence on $S/F(J)$.

Let Ψ be a surjective homomorphism from S to T. Ψ is L' iff s_1 and s_2 are regular elements of S and $\Psi(s_1) = \Psi(s_2)$ implies $s_1 \mathscr{L} s_2$.
Let $S^{L'}$ denote the functorially minimal L' homomorphic image of S.

Lemma 3.2 [8.3.9(c) of [1]]. Let $J_1, \ldots, J_n$ be the regular $\mathscr{J}$-classes of S. Then $S^{L'}$ is isomorphic to a subdirect product of $[S/F(J_i)]/\equiv_R$, $i = 1, 2, \ldots, n$.

Corollary 3.3. Let S be in **SNLG**. Then $S^{L'}$ is an $\mathscr{R}$-trivial semigroup.

Proof. Let J be a regular $\mathscr{J}$-class of S. For any $x \in J$, $s_1, s_2 \in S$. It is easy to see that $x(s_1s_2)^{\omega}s_1 \in F(J)$ iff $x(s_1s_2)^{\omega} \in F(J)$.

If $x(s_1s_2)^{\omega}s_1 \in F(J)$, then $[x(s_1s_2)^{\omega}s_1]\pi = [x(s_1s_2)^{\omega}]\pi$ and so $[x(s_1s_2)^{\omega}s_1]\pi \mathscr{L} [x(s_1s_2)^{\omega}]\pi$ in $S/F(J)$.

If $x(s_1s_2)^{\omega}s_1 \notin F(J)$, then $x(s_1s_2)^{\omega}s_1$, $x(s_1s_2)^{\omega} \in J$. Since J is a left group, $x(s_1s_2)^{\omega}s_1 \mathscr{L} x(s_1s_2)^{\omega}$ in S and so
$[x(s_1s_2)^{\omega}s_1]\pi \mathscr{L} [x(s_1s_2)^{\omega}]\pi$ in $S/F(J)$. It follows that $[(s_1s_2)^{\omega}s_1]\pi \equiv_R [(s_1s_2)^{\omega}]\pi$. Thus the semigroup $[S/F(J)]/\equiv_R$ is $\mathscr{R}$-trivial. Therefore $S^{L'}$ is $\mathscr{R}$-trivial.

In order to state some decompositions of **SNLG**, we need the following notations:

G ··· the variety of groups,
LG ··· the variety of left groups,
R ··· the variety of $\mathscr{R}$-trivial semigroups,
D ··· the variety of semigroups which are nilextensions of left zero semigroups,
SG ··· the variety of Clifford semigroups,
$\mathbf{J}_1$ ··· the variety of semilattices,
LU ··· the variety of semigroups which are nilextensions of left groups.

The main result of the paper is the following theorem.

Theorem 3.4. Let S be a semigroup. Then the following conditions are equivalent:

(1) $S \in [\mathbf{LG}]^{-1}\mathbf{R}$,
(2) $S \in \mathbf{SNLG}$,
(3) $S^{L'} \in \mathbf{R}$,
(4) $S \in \mathbf{G}*\mathbf{R}$,
(5) $S \in \mathbf{D}^{-1}\mathbf{SG}$,
(6) $S \in [\mathbf{LU}]^{-1}\mathbf{J}_1$.

Proof. (1) ⇒ (2) Let Ψ be a surjective homomorphism from S on T with $T \in \mathbf{R}$ such that $e\Psi^{-1} \in \mathbf{LG}$ for all $e \in E(T)$. Then for any $x, y \in S$, $[(yx)^{\omega}y]\Psi = (yx)^{\omega}\Psi$, $[(xy)^{\omega}x]\Psi = (xy)^{\omega}\Psi$ and $(xy)^{\omega}\Psi = (xy)^{\omega+1}\Psi$, since T is $\mathscr{R}$-trivial. It follows that
$[(xy)^{\omega}(yx)^{\omega}]\Psi = [(xy)^{\omega}x(yx)^{\omega}y]\Psi = (xy)^{2\omega+1}\Psi = (xy)^{\omega+1}\Psi = (xy)^{\omega}\Psi$. Hence $(xy)^{\omega}$, $(xy)^{\omega}(yx)^{\omega}$ are in the left group $[(xy)^{\omega}\Psi]\Psi^{-1}$ Since every idempotent of a left group S is a right identity of S,

$$(xy)^{\omega}(yx)^{\omega}\cdot(xy)^{\omega} = (xy)^{\omega}(yx)^{\omega}.$$

Hence $(xy)^{\omega}(yx)^{\omega}$ is an idempotent element and so it is a right identity of $[(xy)^{\omega}\Psi]\Psi^{-1}$. It follows that

$$(xy)^{\omega}(yx)^{\omega} = (xy)^{\omega}\cdot(xy)^{\omega}(yx)^{\omega} = (xy)^{\omega}.$$

Thus $S \in \mathbf{SNLG}$. By Lemma 3.1, $[\mathbf{LG}]^{-1}\mathbf{R} \subseteq \mathbf{SNLG}$.

(2) ⇒ (3) See Corollary 3.3;

(3) ⇒ (4) Since **R** is closed under semidirect products, by Proposition 3.3 of [6], $S \in \mathbf{G}*\mathbf{R}$ iff $S^{L'} \in \mathbf{R}$. Hence (3) implies (4);

(4) ⇒ (1) It is sufficient to show that if S is a semidirect product of a group G with an $\mathscr{R}$-trivial semigroup R, then $S \in [\mathbf{LG}]^{-1}\mathbf{R}$. Let Ψ be the projection from $G*R$ to R. For each $e \in E(R)$,
$e\Psi^{-1} = \{(g, e) : g \in G\}$. Let $(g_1, e), (g_2, e) \in e\Psi^{-1}$. Then $(g_1({}^{e}g_2)^{-1}, e)\cdot(g_2, e) = (g_1({}^{e}g_2)^{-1}({}^{e}g_2), e\cdot e) = (g_1, e)$. It follows that $e\Psi^{-1}$ is a left group.
Hence $S \in [\mathbf{LG}]^{-1}\mathbf{R}$.

(2) ⇒ (5) Let S be in **SNLG**. Then $S = \bigcup_{\alpha \in Y} S_{\alpha}$ with S_{α} being a nilextension of a left group T_{α} for each $\alpha \in Y$. Let ρ be the Clifford congruence on S defined in Proposition 2.2. Then for each $e \in E(S_{\alpha})$, $e\rho = \{s \in S_{\alpha} : es = e\}$, and $E(S_{\alpha}) \subseteq e\rho$. Clearly, $E(e\rho) = E(S_{\alpha})$ is a left zero semigroup and an ideal of $e\rho$. Hence $e\rho$ is a nilextension of a left zero semigroup and so $e\rho \in \mathbf{D}$. Thus $S \in \mathbf{D}^{-1}\mathbf{SG}$;

(5) ⇒ (2) By Lemma 3.1, it is sufficient to show that if there exists a Clifford congruence ρ on S with the property that $e\rho \in \mathbf{D}$ for each $e \in E(S)$, then $S \in \mathbf{SNLG}$. For each $x, y \in S$, We got $(xy)^{\omega}\rho = (yx)^{\omega}\rho$ since S/ρ is a Clifford semigroup and since both $(xy)^{\omega}\rho$ and $(yx)^{\omega}\rho$

are idempotents in the same group, and so $(yx)^{\omega}$, $(xy)^{\omega}$ are in the semigroup $(xy)^{\omega}\rho$. But $(xy)^{\omega}\rho$ is a nilextension of a left zero semigroup. Hence $(xy)^{\omega}=(xy)^{\omega}(yx)^{\omega}$. Thus $S \in$ **SNLG**.

(2) $\Leftrightarrow$ (6) follows from the definition of **SNLG**.

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