An explicit description of positive Riesz distributions on homogeneous cones

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Introduction. Riesz distributions, originally introduced by M. Riesz [6] on the Lorentz cone, are the analytic continuation of the distribution defined by a relatively invariant measure on a homogeneous cone. In general, Riesz distributions are compositions of complex measures supported by the closure of the cone with differential operators. Gindikin [3] describes when the Riesz distribution is a positive measure. Since positive Riesz distributions on symmetric cones are closely connected with the analytic continuation of holomorphic discrete series representations of semisimple Lie groups as is shown by Vergne and Rossi [8], one can determine the so-called Wallach set from the result of Gindikin (see [1, p. 288] for this). Therefore we shall call the positivity set for the Riesz distributions *the Gindikin-Wallach set*. In the present paper, we show that the structure of the Gindikin-Wallach set can be understood clearly by relating it to the orbit structure of the closure of the cone. Moreover we give an explicit description to each of the positive Riesz distributions as a measure on an orbit in the closure of the cone.

1. Preliminaries. Our study is based on the structure theory of normal j-algebras developed in [5]. Here we recall the definition of normal j-algebras. Let $\mathfrak{g}$ be a real split solvable Lie algebra, j a linear mapping on $\mathfrak{g}$ such that $j^2 = -\mathrm{id}_{\mathfrak{g}}$, ω a linear form on $\mathfrak{g}$. The triple $(\mathfrak{g}, j, \omega)$ is called a *normal j-algebra* if the following (i), (ii) are satisfied: (i) $[Y, Y'] + j[Y, jY'] + j[jY, Y'] - [jY, jY'] = 0$ for all $Y, Y' \in \mathfrak{g}$, (ii) $(Y|Y')_\omega := \langle [Y, jY'], \omega\rangle$ defines a j-invariant inner product on $\mathfrak{g}$. We assume throughout this paper that our normal j-algebra $(\mathfrak{g}, j, \omega)$ corresponds to a Siegel domain of tube type. Let $\mathfrak{a}$ be the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ relative to $(\cdot|\cdot)_\omega$. Then $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{g}$. Let $r := \dim \mathfrak{a}$.

Proposition 1 ([5, Chapter 1, Sections 3 and 5]). (i) *There is a linear basis* $\{A_1, \dots, A_r\}$ *of* $\mathfrak{a}$ *such that if one puts* $E_l := -jA_l$, *then* $[A_k, E_l] = \delta_{kl}E_l$ $(1 \le k, l \le r)$.
(ii) *Let* $\alpha_1, \dots, \alpha_r$ *be the basis of* $\mathfrak{a}^*$ *dual to* $A_1, \dots, A_r$. *Then* $\mathfrak{g} = \mathfrak{h} \oplus V$ *with*

$$(1) \qquad \mathfrak{h} = \mathfrak{a} \oplus \Big(\sum_{1\le k<m\le r}^{\oplus} \mathfrak{g}_{(\alpha_m-\alpha_k)/2} \Big),$$

$$(2) \quad V = \Big(\sum_{k=1}^{r}{}^{\oplus} \mathbf{R}E_k \Big) \oplus \Big(\sum_{1\le k<m\le r}^{\oplus} \mathfrak{g}_{(\alpha_m+\alpha_k)/2} \Big),$$

where $\mathfrak{g}_\alpha := \{Y \in \mathfrak{g} \mid [C, Y] = \alpha(C)Y$ *for all* $C \in \mathfrak{a}\}$ *for* $\alpha \in \mathfrak{a}^*$.
(iii) *One has* $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, V] \subset V$, *and* $[V, V] = \{0\}$.

Let H be the simply connected Lie group corresponding to $\mathfrak{h}$. Then H acts on V by the adjoint action.

Lemma 2 ([7, Theorem 4.15]). *Put* $E := \sum_{k=1}^r E_k \in V$ *and let* Ω *be the* H*-orbit in* V *through* E. *Then* Ω *is an open convex cone in* V *containing no line, and* H *acts on* Ω *simply transitively.*

According to (1), we express every $T \in \mathfrak{h}$ by $T = \sum_{k=1}^r t_{kk}A_k + \sum_{m>k} T_{mk}$ $(t_{kk} \in \mathbf{R}, T_{mk} \in \mathfrak{g}_{(\alpha_m-\alpha_k)/2})$. Let Π be the open subset $\{T \in \mathfrak{h} \mid t_{kk} > 0$ for all $k = 1, \dots, r\}$ of $\mathfrak{h}$. Putting $T_{kk} := (2\log t_{kk})A_k$ $(1 \le k \le r)$ and $L_k := \sum_{m>k} T_{mk}$ $(1 \le k \le r-1)$ for $T \in \Pi$, we set $\gamma(T) := \exp T_{11} \cdot \exp L_1 \cdot \exp T_{22} \cdots \exp L_{r-1} \cdot \exp T_{rr}$. Then γ is a diffeomorphism from Π onto H. Using this γ, we have the following multiplication formula for the elements of H.

Proposition 3. *For* $T, T' \in \Pi$, *one has* $\gamma(T)\gamma(T') = \gamma(T'')$ *with*

$$t''_{kk} = t_{kk}t'_{kk} \quad (1 \le k \le r),$$
$$T''_{mk} = t_{mm}T'_{mk} + \sum_{k<l<m} [T_{ml}, T'_{lk}] + t'_{kk}T_{mk} \quad (1 \le k < m \le r).$$

2. Orbit structure of $\overline{\Omega}$. Accordind to (2), we decompose every $x \in V$ as $x = \sum_{k=1}^r x_{kk}E_k + \sum_{m>k} X_{mk}$ $(x_{kk} \in \mathbf{R}, X_{mk} \in \mathfrak{g}_{(\alpha_m+\alpha_k)/2})$. Define $E^* \in \mathfrak{g}^*$ by $\langle x + T, E^*\rangle = \sum_{k=1}^r x_{kk}$ $(x \in V,$

$T \in \mathfrak{h}$) and set $(Y|Y') := \langle [jY, Y'], E^* \rangle /2$ ($Y, Y' \in \mathfrak{g}$). Then $(\cdot|\cdot)$ defines a new inner product on $\mathfrak{g}$ and we shall call it *the standard inner product* on $\mathfrak{g}$. For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \{0, 1\}^r$, put $E_\varepsilon := \sum_{k=1}^r \varepsilon_k E_k \in V$. Let $\mathcal{O}_\varepsilon$ be the H-orbit in V through E_ε. Note that if $\varepsilon = \mathbf{1} := (1, \ldots, 1)$, then $E_\varepsilon = E$ and $\mathcal{O}_\varepsilon = \Omega$.

Proposition 4. *Let* $x := \gamma(T) \cdot E_\varepsilon \in \mathcal{O}_\varepsilon$ ($T \in \Pi$). *Then*

$$x_{kk} = \varepsilon_k (t_{kk})^2 + \sum_{i<k} \varepsilon_i (T_{ki} \mid T_{ki}) \ (1 \le k \le r),$$

$$X_{mk} = \varepsilon_k t_{kk} [T_{mk}, E_k] + \sum_{i<k} \varepsilon_i [T_{mi}, [T_{ki}, E_i]] \quad (1 \le k < m \le r).$$

One can deduce the following two lemmas from Proposition 4.

Lemma 5. *Let H_{E_ε} be the stabilizer in H at E_ε. Then $H_{E_\varepsilon} = \{\gamma(T) \in H \mid$ if $\varepsilon_i = 1$, then $t_{ii} = 1$ and $T_{ki} = 0$ $(k > i)\}$.*

Lemma 6. *Put $H(\mathcal{O}_\varepsilon) := H_{E_{1-\varepsilon}}$. Then $H(\mathcal{O}_\varepsilon)$ acts simply transitively on the H-orbit $\mathcal{O}_\varepsilon$.*

Now the H-orbit decomposition of the closure $\bar{\Omega}$ of Ω is as follows:

Theorem 7. $\bar{\Omega} = \bigsqcup_{\varepsilon \in \{0,1\}^r} \mathcal{O}_\varepsilon$.

3. Γ-integrals and Riesz distributions. For $s = (s_1, \ldots, s_r) \in \mathbf{C}^r$, we define a character χ_s on H by $\chi_s(\gamma(T)) := (t_{11})^{2s_1} \cdots (t_{rr})^{2s_r}$ ($T \in \Pi$). Set $C(\varepsilon) := \{s \in \mathbf{C}^r \mid s_i = 0$ for all i such that $\varepsilon_i = 0\}$. From Lemma 5 we see that $\chi_s = 1$ on H_{E_ε} if and only if $s \in C(\varepsilon)$. Thus for $s \in C(\varepsilon)$, we can define a function Δ_s^ε on $\mathcal{O}_\varepsilon$ by $\Delta_s^\varepsilon(t \cdot E_\varepsilon) := \chi_s(t)$ ($t \in H$). Clearly

$$\Delta_s^\varepsilon(t_0 \cdot x) = \chi_s(t_0)\, \Delta_s^\varepsilon(x) \ (t_0 \in H,\ x \in V).$$

For $s \in \mathbf{C}^r$, we set $\varepsilon \cdot s := (\varepsilon_1 s_1, \ldots, \varepsilon_r s_r)$. Then $\varepsilon \cdot s \in C(\varepsilon)$. Let $p(\varepsilon) = (p_1(\varepsilon), \ldots, p_r(\varepsilon)) \in \mathbf{Z}^r$ be given by $p_k(\varepsilon) := \sum_{i<k} \varepsilon_i \dim \mathfrak{g}_{(\alpha_k - \alpha_i)/2}$ ($1 \le k \le r$). When $\varepsilon \ne \mathbf{0} := (0, \ldots, 0)$, keeping Lemma 6 in mind, we define a measure μ_ε on $\mathcal{O}_\varepsilon$ by

$$d\mu_\varepsilon(t \cdot E_\varepsilon) := \chi_{\varepsilon \cdot (1 + p(\varepsilon))/2}(t) \prod_{\varepsilon_i = 1} dt_{ii} \prod_{\varepsilon_i = 1, k > i} dT_{ki} \ (t \in H(\mathcal{O}_\varepsilon)),$$

where dT_{ki} stands for the Euclidean measure on $\mathfrak{g}_{(\alpha_k - \alpha_i)/2}$ normalized by the standard inner product on $\mathfrak{g}$. Let μ_0 be the Dirac measure at $x = 0$.

Lemma 8. *The measure μ_ε is relatively invariant under H:*

$$d\mu_\varepsilon(t_0 \cdot x) = \chi_{(1-\varepsilon) \cdot p(\varepsilon)/2}(t_0)\, d\mu_\varepsilon(x) \ (t_0 \in H).$$

In particular, μ_ε is $H(\mathcal{O}_\varepsilon)$-invariant.

This lemma states that μ_ε is the transfer of the left Haar measure on $H(\mathcal{O}_\varepsilon)$ by the orbit map $H(\mathcal{O}_\varepsilon) \ni t \mapsto t \cdot E_\varepsilon \in \mathcal{O}_\varepsilon$.

Now we consider the following Γ-integral on the orbit $\mathcal{O}_\varepsilon$:

(3) $\Gamma_{\mathcal{O}_\varepsilon}(s) := \int_{\mathcal{O}_\varepsilon} e^{-\langle x, E^* \rangle} \Delta_s^\varepsilon(x)\, d\mu_\varepsilon(x)$ ($s \in C(\varepsilon)$).

Theorem 9. *The integral* (3) *converges if and only if*

(4) $\mathfrak{R} s_i > p_i(\varepsilon)/2$ *for all i such that $\varepsilon_i = 1$.*

Moreover, when this condition is satisfied, one has

$$\Gamma_{\mathcal{O}_\varepsilon}(s) = 2^{-|\varepsilon|} \pi^{|p(\varepsilon)|/2} \prod_{\varepsilon_i = 1} \Gamma\left(s_i - \frac{p_i(\varepsilon)}{2}\right),$$

where $|\varepsilon| := \sum_{i=1}^r \varepsilon_i$ *and* $|p(\varepsilon)| := \sum_{i=1}^r p_i(\varepsilon)$.

When $\varepsilon = \mathbf{1}$, this theorem yields Gindikin's formula [2, Theorem 2.1].

Let $D(\varepsilon) := C(\varepsilon) + (\mathbf{1} - \varepsilon) \cdot p(\varepsilon)/2$. Then

$$\Delta_{\varepsilon \cdot s}^\varepsilon(t_0 \cdot x)\, d\mu_\varepsilon(t_0 \cdot x) = \chi_s(t_0)\, \Delta_{\varepsilon \cdot s}^\varepsilon(x)\, d\mu_\varepsilon(x)$$

($s \in D(\varepsilon)$, $t_0 \in H$, $x \in \mathcal{O}_\varepsilon$).

Let $\Xi_{\mathbf{C}}(\varepsilon)$ be the set of $s \in D(\varepsilon)$ such that $\varepsilon \cdot s \in C(\varepsilon)$ satisfies (4).

Theorem 10. (i) *For any rapidly decreasing function φ on V and any $s \in \Xi_{\mathbf{C}}(\varepsilon)$, the following integral converges*:

$$\langle \mathcal{R}_s^\varepsilon, \varphi \rangle := \frac{1}{\Gamma_{\mathcal{O}_\varepsilon}(\varepsilon \cdot s)} \int_{\mathcal{O}_\varepsilon} \varphi(x)\, \Delta_{\varepsilon \cdot s}^\varepsilon(x)\, d\mu_\varepsilon(x).$$

(ii) *$\langle \mathcal{R}_s^\varepsilon, \varphi \rangle$ admits an analytic continuation as a holomorphic function of $s \in D(\varepsilon)$, and defines a tempered distribution.*

This theorem also contains [2, Theorem 3.1] as the special case $\varepsilon = \mathbf{1}$. We simply write $\mathcal{R}_s$ for $\mathcal{R}_s^1$, and call it *the Riesz distribution on Ω*. Since $D(1) = \mathbf{C}^r$, $\mathcal{R}_s$ is defined for all $s \in \mathbf{C}^r$.

Proposition 11. (i) *For $s \in D(\varepsilon)$ and $t \in H$, one has* $\langle \mathcal{R}_s^\varepsilon, e^{-\langle t \cdot x, E^* \rangle} \rangle_x = \chi_s(t^{-1})$.

(ii) *The distribution $\mathcal{R}_s^\varepsilon$ coincides with $\mathcal{R}_s$ for any $\varepsilon \in \{0, 1\}^r$ and $s \in D(\varepsilon)$.*

Let $\Xi(\varepsilon) := \Xi_{\mathbf{C}}(\varepsilon) \cap \mathbf{R}^r$. Then we see that $\Xi(\varepsilon)$ is the set of all $s \in \mathbf{R}^r$ such that $s_k = p_k(\varepsilon)/2$ (if $\varepsilon_k = 0$), $s_k > p_k(\varepsilon)/2$ (if $\varepsilon_k = 1$). These 2^r sets $\Xi(\varepsilon)$'s are mutually disjoint. The following theorem is an immediate consequence of Theorem 10 (i) and Proposition 11 (ii).

Theorem 12. *If $s \in \Xi(\varepsilon)$, then $\mathcal{R}_s$ is a positive measure on the H-orbit $\mathcal{O}_\varepsilon$ described as* $d\mathcal{R}_s = \Gamma_{\mathcal{O}_\varepsilon}(\varepsilon \cdot s)^{-1} \Delta_{\varepsilon \cdot s}^\varepsilon\, d\mu_\varepsilon$.

Furthermore all of the positive Riesz distributions are obtained in this way:

Theorem 13. *The Riesz distribution $\mathcal{R}_s$ is positive if and only if s belongs to the union* $\bigsqcup_{\varepsilon \in \{0,1\}^r} \Xi(\varepsilon)$ $(=: \Xi)$.

Thus the Gindikin-Wallach set is equal to

Ξ and decomposed into 2^r subsets $\Xi(\varepsilon)$ according to the support of the corresponding Riesz distributions.

We conclude this paper by introducing an algorithm to determine whether a given $s \in \mathbf{R}^r$ belongs to Ξ or not. Moreover, if $s \in \Xi$, the algorithm gives the ε for which $s \in \Xi(\varepsilon)$. For $s \in \mathbf{R}^r$, define $\sigma^{[1]}, \ldots, \sigma^{[r]} \in \mathbf{R}^r$ by $\sigma^{[1]} := s$ and
$$\sigma^{[k+1]} := \begin{cases} \sigma^{[k]} - (0, \ldots, 0, n_{k+1,k}/2, \ldots, n_{rk}/2) & (\sigma_k^{[k]} > 0), \\ \sigma^{[k]} & (\sigma_k^{[k]} = 0), \end{cases}$$
for $k = 1, \ldots, r-1$, where $n_{mk} := \dim \mathfrak{g}_{(\alpha_m - \alpha_k)/2}$ $(m > k)$.

Proposition 14. (i) *The parameter $s \in \mathbf{R}^r$ belongs to Ξ if and only if $\sigma_k^{[k]} \geq 0$ for all $k = 1, \ldots, r$.*
(ii) *When $s \in \Xi$, let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ be the element of $\{0, 1\}^r$ given by*
$$\varepsilon_k := 1 \ (\text{if } \sigma_k^{[k]} > 0), \quad \varepsilon_k := 0 \ (\text{if } \sigma_k^{[k]} = 0).$$
Then $s \in \Xi(\varepsilon)$.

The details [4] will appear elsewhere.

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