# On the $\lambda$-invariants of totally real fields

By Jangheon OH*)

KIAS, 207-43 Cheongryangri-dong, Dongdaemun-gu, Seoul 130-012, Korea

(Communicated by Shokichi IYANAGA, M. J. A., Oct. 12, 1998)

**1. Introduction.** Let $k$ be a number field and $p$ be a prime number, and let $k = k_0 \subset k_1 \subset \cdots \subset k_n \subset \cdots \subset k_\infty$ be the cyclotomic $\boldsymbol{Z}_p$-extension of $k$. We denote by $\mu_p(k)$, $\lambda_p(k)$ the Iwasawa invariants of the cyclotomic $\boldsymbol{Z}_p$-extension of $k$. It is well-known that $\mu_p(k)$ vanishes for any abelian number field $k$. Greenberg's conjecture claims that both $\mu_p(k)$ and $\lambda_p(k)$ are zero for any totally real number field $k$. In this paper, we shall prove the following

**Theorem 1.** *Let $p$ and $q$ be prime numbers such that $p \equiv 3 \bmod 8$, $q \equiv -1 \bmod 8$, $p \not\equiv 3 \bmod 16$, $q \not\equiv -1 \bmod 16$. Then the Iwasawa invariant $\lambda_2(\boldsymbol{Q}(\sqrt{pq}))$ is zero. Let $p$, $q$ and $r$ be prime numbers such that $p, q \equiv 3 \bmod 8$, $p, q \not\equiv 3 \bmod 16$. $r \equiv 1 \bmod 4$, $r \not\equiv 1 \bmod 8$. Then the Iwasawa invariant $\lambda_2(\boldsymbol{Q}(\sqrt{pqr}))$ is zero if there is no element $\alpha$ in the unit group of $k_1 = \boldsymbol{Q}(\sqrt{pqr}, \sqrt{2})$ such that $N_{k_1/\mathrm{Q}_1}\alpha = -1$.*

Let $p$ and $\ell$ be odd prime numbers such that $p \equiv 1 \bmod \ell$. Let $k$ be a subfield of degree $\ell$ of $\boldsymbol{Q}(\zeta_{p\ell^2})$ in which $p$ and $\ell$ ramify. Here $\zeta_{p\ell^2}$ is a primitive $p\ell^2$-th root of unity. We will prove the following

**Theorem 2.** *Let $p$ and $\ell$ be odd prime numbers such that $p \equiv 1 \bmod \ell$, $p \not\equiv 1 \bmod \ell^2$. Then the Iwasawa invariants $\mu_\ell(k)$ and $\lambda_\ell(k)$ vanish, where $k$ is the number field constructed above.*

Now let $p$ be a prime number and $k$ be a totally real number field and $K$ be a real cyclic extension of degree $p$ over $k$, which satisfies $K \cap k_\infty = k$. Let $S_{K_\infty/k_\infty} = \{w$: prime ideal of $K_\infty \mid w$ is prime to $p$ and ramified in $K_\infty/k_\infty\}$.

In [1], Iwasawa proved a "plus-version" of Kida's formula. In [2], the following theorem is obtained by using the above Iwasawa's formula.

**Theorem 3.** *Let $p$ be a prime number, $k$ a totally real number field of finite degree and $K$ a real cyclic extension of degree $p$ over $k$. Assume that $k_\infty$ has only one prime ideal lying over $p$ and that the class number of $k$ is not divisible by $p$. Then, the following are equivalent:*

(1) $\lambda_p(K) = 0$.

(2) *For any prime ideal $w$ of $K_\infty$ which is prime to $p$ and ramified in $K_\infty/k_\infty$, the order of ideal class of $w$ is prime to $p$.*

In this paper, we apply Theorem 3 to prove Theorem 1 and Theorem 2. We state another ingredient needed here. Let $K$ be a cyclic extension of a number field F. Let $G = Gal(K/F)$. For each valuation $v$ of $F$ we let $e(v)$ be the ramification index of $v$ in $K/F$. We put $e(K/F) = \Pi_v e(v)$. We let $E_K$ denote the group of units, $C_K$ the group of ideal classes, $C_K^G$ the set of ambiguous ideal class groups, and $C_K'^G$ the set of ideal class groups containing ambiguous ideal of $K$, respectively. We will use the following "genus formula":

**Theorem 4.** *Let $K/F$ be a cyclic extension with Galois group $G$. Then*

$$(1)\quad |C_K^G| = \frac{h(F)e(K/F)}{[K:F](E_F : N_{K/F}K^* \cap E_F)}.$$

$$|C_K'^G| = \frac{h(F)e(K/F)}{[K:F](E_F : N_{K/F}E_K)}.$$

*Proof.* See [3, p. 307]. □

**2. Proof of theorems.** Before proving Theorem 1, we need the following

**Lemma 1.** *Let $D$ be a square free positive integer such that there exists a prime number $q \mid D$ such that $q \equiv -1 \bmod 8$. Let $k = \boldsymbol{Q}(\sqrt{D})$. Then there is no element $\alpha$ in the first layer $k_1$ in the cyclotomic $\boldsymbol{Z}_2$-extension of $k$ such that*

$$N_{k_1/\mathrm{Q}_1}(\alpha) = -1.$$

*Proof.* First note that $(\frac{-1}{q}) = -1$ and $(\frac{2}{q}) = 1$. Suppose that there is an $\alpha$ in $k_1$ such that

$$(2)\qquad N_{k_1/\mathrm{Q}_1}(\alpha) = -1.$$

Write $\alpha = x + y\sqrt{2} + z\sqrt{D} + w\sqrt{2D}$, where $x$, $y$, $z$ and $w$ are in $\boldsymbol{Q}$.

Then by (2) we have

*) Supported by KIAS. I would like to thank Prof. K. Komatsu for reading this paper and giving many valuable comments.

(3) $(x + y\sqrt{2})^2 - D(z + w\sqrt{2})^2 = -1$

Clearing the denominators of (3), we have

(4) $a^2 + 2b^2 + m^2 = D(c^2 + 2d^2)$, $ab = Dcd$

for some integers $a, b, c, d$ and $m$. If $q$ divides $m$, we see that $q$ divides $a$ and $b$ since $q$ divides $m$. Since $D$ is square free, we see that $q$ divides $c$ and $d$. Hence we may assume that $q$ is relatively prime to $m$. Reducing both sides of (4) by mod $q$, we have

(5) $a^2 + 2b^2 + m^2 \equiv 0$, $ab \equiv 0 \bmod q$.

If $a \equiv 0 \bmod q$, then we have $m^2 + 2b^2 \equiv 0 \bmod q$. This is a contradiction since $-2$ is not a square mod $q$. If $b \equiv 0 \bmod q$, then we have $a^2 + m^2 \equiv 0 \bmod q$. This is also a contradiction since $-1$ is not a square mod $q$. This completes the proof. □

**Lemma 2.** *Let $D$ be a square free positive integer such that there exist a prime number $p \mid D$ such that $p \equiv 3$ mod $8$. Let $k = \boldsymbol{Q}(\sqrt{D})$. Then there is no $\alpha$ in $k_1$ such that*

$$N_{k_1/Q_1}(\alpha) = \pm (\sqrt{2} - 1).$$

*Proof.* We omit the proof since the proof is similar to Lemma 1.

**Proof of Theorem 1.** First we prove the first part of Theorem 1. By assumptions on $p$ and $q$, we have

(6) $S_{k_\infty/Q_\infty} = \{\mathfrak{p}, \mathfrak{q}_1, \mathfrak{q}_2\}$,

where $\mathfrak{p}$ is the prime ideal of $k_1$ lying over $p$ and $\mathfrak{q}_1$, $\mathfrak{q}_2$ are prime ideals of $k_1$ lying over $q$. Note that $E_{Q_1} = \langle \pm 1 \rangle (\sqrt{2} - 1)^{\mathbf{Z}}$. Hence $e(k_1/\boldsymbol{Q}_1) = 8$ and $[E_{Q_1} : N_{k_1/Q_1}k_1^* \cap E_{Q_1}] = 4$ by Lemma 1 and 2. This completes the proof of the first part by Theorem 3 and Theorem 4.

Now let $k = \boldsymbol{Q}(\sqrt{pqr})$, where $p, q$ and $r$ are prime numbers such that $p, q \equiv 3 \bmod 8$, $p, q \not\equiv 3 \bmod 16$. $r \equiv 1 \bmod 4$, $r \not\equiv 1 \bmod 8$. By these assumptions on $p, q$ and $r$, we have

$$S_{k_\infty/Q_\infty} = \{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}\}.$$

Our conclusion follows immediately from Lemma 2, Theorem 3 and Theorem 4. □

**Remark 1.** *Actually the prime ideals $\mathfrak{p}, \mathfrak{q}_1, \mathfrak{q}_2$ of $k_1$ are principal. Let $p$ and $q$ be prime numbers such that $p, q \equiv 3$ mod $8$, $p, q \not\equiv 3$ mod $16$. Then we can prove similarly that the Iwasawa invariants $\lambda_2(\boldsymbol{Q}(\sqrt{p}))$ and $\lambda_2(\boldsymbol{Q}[\sqrt{pq}))$ are zero,* [5] *contains another proof of this. It can be shown that there always exists an $\alpha$ in $k_1$ such that $N_{k_1/Q_1}(\alpha) = -1$.*

**Example 1.** *Let $k = \boldsymbol{Q}[\sqrt{5 * 11 * 43})$, or $\boldsymbol{Q}(\sqrt{37 * 11 * 43})$. By using number theoretic packages "KASH", we can see that there is no unit $\alpha$ in $k_1$ such that $N_{k_1/Q_1}(\alpha) = -1$. Hence $\lambda_2(k) = 0$.*

**Example 2.** *Let $k = \boldsymbol{Q}(\sqrt{37 * 59 * 43})$. Again, by using KASH, we see that there is a unit $\alpha$ in $k_1$ such that $N_{k_1/Q_1}(\alpha) = -1$. In this case, we can not decide whether $\lambda_2(k)$ is zero or not. Note that the class numbers of $k$ and $k_1$ are 2 and 8, respectively.*

**Proof of Theorem 2.** Note that $S_{k_\infty/Q_\infty} = \{\mathfrak{p}\}$. Let $\ell_1$ and $\mathfrak{p}_1$ be prime ideals of $k_1$ above $\ell$ and $p$, respectively. We see that $\ell_1$ is unramified in the extension $k_1/\boldsymbol{Q}_1$ since $k_1/k$ is unramified everywhere. Hence $\mathfrak{p}_1$ is principal in $k_1$ by the genus formula. This completes the proof of Theorem 2. □

## References


[1] K. Iwasawa: Riemann-Hurwitz formula and $p$-adic Galois representation for number fields. Tôhoku Math. J., **33**, 263–288 (1981).

[2] T. Fukuda, K. Komatsu, M. Ozaki, and H. Taya: On Iwasawa $\lambda_p$-invariants of relative real cyclic extensions of degree $p$, Tokyo J. Math., **20**, 489–494 (1997).

[3] S. Lang: Cyclotomic Fields I and II. Graduate Texts in Mathematics, Springer-Verlag, New York (1990).

[4] S. Lang: Algebraic Number Theory. Graduate Texts in Mathematics, Springer-Verlag, New York (1986).

[5] M. Ozaki and H. Taya: On the Iwasawa $\lambda_2$-invariants of certain families of real quadratic fields. Manuscripta Math., **94**, 437–444 (1997).