On the rank of the elliptic curve $y^2 = x^3 + kx$

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In this note, we consider the elliptic curve

$$\varepsilon_k : y^2 = x^3 + kx.$$

Mestre showed in [2] that there are infinitely many values of $k \in \boldsymbol{Q}$, for which the rank of ε_k is at least 4. Nagao showed the same result in [3] by a different construction. We shall improve this result in this paper.

(See Theorem 2 below.)

Let $k(t) = -16(-2+t^2)^2(2+2t+t^2)^2$ $(6+4t+t^2)(2+4t+3t^2)*$ $(16+56t+86t^2+68t^3+31t^4+8t^5+t^6)*$ $(8+32t+62t^2+68t^3+43t^4+14t^5+2t^6)*$ $(28+136t+296t^2+368t^3+287t^4+146t^5+49t^6+10t^7+t^8)*$ $(32+176t+460t^2+680t^3+612t^4+340t^5+115t^6+22t^7+2t^8)*$ $(48+224t+492t^2+656t^3+572t^4+328t^5+123t^6+28t^7+3t^8)*$ $(64+320t+784t^2+1168t^3+1148t^4+736t^5+296t^6+68t^7+7t^8)$.

We consider the following elliptic curve

$\varepsilon_{k(t)} : y^2 = x^3 + k(t)x$

$\varepsilon_{k(t)}$ have 5 $\boldsymbol{Q}(t)$-rational points $P_i = (x_i, y_i)$ $(1 \le i \le 5)$, where

$x_1 = -4(-2+t^2)^2(2+2t+t^2)^4(6+4t+t^2)$ $(2+4t+3t^2)*$ $(16+56t+86t^2+68t^3+31t^4+8t^5+t^6)*$ $(8+32t+62t^2+68t^3+43t^4+14t^5+2t^6)$,

$y_1 = 8(-2+t^2)^2(2+2t+t^2)^3(6+4t+t^2)(2+4t+3t^2)*$ $(16+56t+86t^2+68t^3+31t^4+8t^5+t^6)*$ $(8+32t+62t^2+68t^3+43t^4+14t^5+2t^6)*$ $(1536+15872t+78656t^2+246720t^3+546512t^4+904288t^5+1153680t^6+1155360t^7+916600t^8+577680t^9+288420t^{10}+113036t^{11}+34157t^{12}+7710t^{13}+1229t^{14}+124t^{15}+6t^{16})$,

$x_2 = -4(-2+t^2)(6+4t+t^2)(2+4t+3t^2)*$ $(16+56t+86t^2+68t^3+31t^4+8t^5+t^6)*$ $(28+136t+296t^2+368t^3+287t^4+146t^5+49t^6+10t^7+t^8)*$ $(32+176t+460t^2+680t^3+612t^4+340t^5+115t^6+22t^7+2t^8)$,

$y_2 = 8(-2+t^2)^2(6+4t+t^2)(2+4t+3t^2)*$ $(16+56t+86t^2+68t^3+31t^4+8t^5+t^6)*$ $(28+136t+296t^2+368t^3+287t^4+146t^5+49t^6+10t^7+t^8)*$ $(32+176t+460t^2+680t^3+612t^4+340t^5+115t^6+22t^7+2t^8)*$ $(192+1696t+6840t^2+16704t^3+27476t^4+32080t^5+27318t^6+17168t^7+7947t^8+2658t^9+613t^{10}+88t^{11}+6t^{12})$,

$x_3 = (-2+t^2)(6+4t+t^2)(2+4t+3t^2)*$ $(8+32t+62t^2+68t^3+43t^4+14t^5+2t^6)*$ $(32+176t+460t^2+680t^3+612t^4+340t^5+115t^6+22t^7+2t^8)*$ $(64+320t+784t^2+1168t^3+1148t^4+736t^5+296t^6+68t^7+7t^8)$,

$y_3 = (-2+t^2)^2(6+4t+t^2)(2+4t+3t^2)*$ $(8+32t+62t^2+68t^3+43t^4+14t^5+2t^6)*$ $(32+176t+460t^2+680t^3+612t^4+340t^5+115t^6+22t^7+2t^8)*$ $(64+320t+784t^2+1168t^3+1148t^4+736t^5+296t^6+68t^7+7t^8)*$ $(768+5632t+19616t^2+42528t^3+63576t^4+68672t^5+54636t^6+32080t^7+13738t^8+4176t^9+855t^{10}+106t^{11}+6t^{12})$,

$x_4 = -4(-2+t^2)^2(2+2t+t^2)(2+4t+3t^2)*$ $(48+224t+492t^2+656t^3+572t^4+328t^5+123t^6+28t^7+3t^8)*$ $(64+320t+784t^2+1168t^3+1148t^4+736t^5+296t^6+68t^7+7t^8)$,

$y_4 = 8(-2+t^2)^2(2+2t+t^2)^2(2+4t+3t^2)*$ $(48+224t+492t^2+656t^3+572t^4+328t^5+123t^6+28t^7+3t^8)*$ $(64+320t+784t^2+1168t^3+1148t^4+736t^5+296t^6+68t^7+7t^8)*$ $(576+5120t+21616t^2+56912t^3+103600t^4+137144t^5+135656t^6+101764t^7+58308t^8+25506t^9+8431t^{10}+2054t^{11}+351t^{12}+38t^{13}+2t^{14})$,

$x_5 = -4t^4(-2+t^2)^2(2+2t+t^2)(6+4t+$

$t^2)*$
$(28 + 136t + 296t^2 + 368t^3 + 287t^4 + 146t^5 + 49t^6 + 10t^7 + t^8)*$
$(48 + 224t + 492t^2 + 656t^3 + 572t^4 + 328t^5 + 123t^6 + 28t^7 + 3t^8)$,
$y_5 = 8t^2(-2 + t^2)^2(2 + 2t + t^2)^2(6 + 4t + t^2)*$
$(28 + 136t + 296t^2 + 368t^3 + 287t^4 + 146t^5 + 49t^6 + 10t^7 + t^8)*$
$(48 + 224t + 492t^2 + 656t^3 + 572t^4 + 328t^5 + 123t^6 + 28t^7 + 3t^8)*$
$(512 + 4864t + 22464t^2 + 65728t^3 + 134896t^4 + 204048t^5 + 233232t^6 + 203528t^7 + 135656t^8 + 68572t^9 + 25900t^{10} + 7114t^{11} + 1351t^{12} + 160t^{13} + 9t^{14})$. (cf. [4])

Now, we have the following theorems.

Theorem 1. *$P_1, \ldots, P_5$ are independent points.*

Proof. We specialize $t = 1$. Then we have 5 rational points $Q_1, \ldots, Q_5$ obtained from $P_1, \ldots, P_5$. By using calculation system PARI, we see that the determinant of the matrix $(< Q_i, Q_j >)(1 \le i, j \le 5)$ associated to the canonical height is 224982.73. Since this determinant is non-zero, we see $P_1, \ldots, P_5$ are independent. Q.E.D.

Theorem 2. *There are infinitely many non isomorphic elliptic curves of the form $y^2 = x^3 + kx$ with rank at least 5 over $\boldsymbol{Q}$.*

Proof. From Theorem 1 and the Theorem 20.3 in [1], there are infinitely many rational values of t, for which the rank of $\varepsilon_{k(t)} \ge 5$. So it suffices to show that there are only a finite number of rational values of t for which $\varepsilon_{k(t)}$ is isomorphic to $\varepsilon_{k(t_0)}$ for a given $t_0 \in \boldsymbol{Q}$. Let $k(t_0) = k_0$. Then the isomorphism of $\varepsilon_{k(t)}$ with ε_{k_0} implies $k(t) = k_0 u^4$, $u \in \boldsymbol{Q}$ or

$$(*) \quad \left(\frac{u^2}{(-2 + t^2)(2 + 2t + t^2)}\right)^2 = \frac{k(t)}{k_0((-2 + t^2)(2 + 2t + t^2))^2} = F(t) \in \boldsymbol{Q}(t),$$

$\deg F(t) = 48$.

The finiteness of the number of rational values of t for which (*) holds follows from Faltings' theorem as $y^2 = F(x)$ is embedded in a smooth rational curve with the genus 23 (cf. [1], p. 44). Q.E.D.

References

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