Gröbner deformations of regular holonomic systems

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1. Torus-fixed ideals in the Weyl algebra. This is a research announcement of results in the first part of our monograph [15]. Let $D = C\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ denote the Weyl algebra with complex coefficients. Thus D is the free associative C-algebra on $2n$ generators modulo the relations $x_i x_j = x_j x_i$, $\partial_i \partial_j = \partial_j \partial_i$, $x_i \partial_j = \partial_j x_i - \delta_{ij}$. Left ideals in D are called *D-ideals*. They represent systems of linear partial differential equations with polynomial coefficients. The torus $(C^*)^n$ acts on the Weyl algebra by $\partial_i \mapsto t_i \partial_i$ and $x_i \mapsto t_i^{-1} x_i$ for $(t_1, \ldots, t_n) \in (C^*)^n$. We abbreviate $\theta_i = x_i \partial_i$. The set of elements in D which are fixed by $(C^*)^n$ equals the commutative polynomial subring $C[\theta] = C[\theta_1, \ldots, \theta_n]$.

Lemma 1.1. *A D-ideal J is torus-fixed if and only if J is generated by (finitely many) elements of the form $x^a \cdot p(\theta) \cdot \partial^b$ where $a, b \in N^n$ and $p(\theta) \in C[\theta]$.*

Each $f \in D$ is written uniquely as a finite sum $f = \sum_{a,b \in N^n} c_{ab} x^a \partial^b$ with $c_{ab} \in C$. Fix $u, v \in R^n$ with $u + v \geq 0$. Then $\mathrm{in}_{(u,v)}(f) \in D$ is the subsum of all terms $c_{ab} x^a \partial^b$ for which $u \cdot a + v \cdot b$ is maximal. For a D-ideal I we define the *initial ideal* $\mathrm{in}_{(u,v)}(I)$ to be the C-vector space spanned by $\{\mathrm{in}_{(u,v)}(f) : f \in I\}$. If $u + v > 0$ then $\mathrm{in}_{(u,v)}(I)$ is generally not a D-ideal; it is an ideal in the commutative polynomial ring $gr(D) = C[x, \xi] = C[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$. Generators for the initial ideal can be computed by the Weyl algebra version of Buchberger's Gröbner basis algorithm; see e.g. [3] and [6] for early treatments and [13] for a precise introduction and recent applications.

If $u + v = 0$ then the initial ideal is a D-ideal. For $w \in R^n$ we call $\mathrm{in}_{(-w,w)}(I)$ a *Gröbner deformation* of I. Specifically, if $w \in Z^n$ then the D-ideal $\mathrm{in}_{(-w,w)}(I)$ is regarded as the limit of I under the one-parameter subgroup of $(C^*)^n$ defined by w.

Lemma 1.2. *For generic $w \in R^n$, the initial D-ideal $\mathrm{in}_{(-w,w)}(I)$ is torus-fixed.*

Let $D^{\pm} := C\langle x_1^{\pm 1}, \ldots, x_n^{\pm 1}, \partial_1, \ldots, \partial_n\rangle$ be the ring of differential operators on $(C^*)^n$. For a D-ideal I define the commutative polynomial ideal $\tilde{I} := D^{\pm} I \cap C[\theta]$.

Proposition 1.3. *If J is a torus-fixed D-ideal then $\tilde{J} \subset C[\theta]$ is generated by $p(\theta - b) \cdot \prod_{i=1}^{n} \prod_{j=1}^{b_i} (\theta_i + 1 - j)$ where $x^a \cdot p(\theta) \cdot \partial^b$ runs over a generating set of J.*

2. Holonomic rank under Gröbner deformations. Abbreviate $e := (1, 1, \ldots, 1) \in R^n$. The ideal $\mathrm{in}_{(0,e)}(I)$ in $C[x, \xi]$ is called the *characteristic ideal* of the D-ideal I. The *Fundamental Theorem of Algebraic Analysis* ([5],[12],[14]) states that each minimal prime of the characteristic ideal $\mathrm{in}_{(0,e)}(I)$ has dimension $\geq n$. If $\mathrm{in}_{(0,e)}(I)$ has dimension n then I is *holonomic*. In this case the following vector space dimension is finite and is called the *holonomic rank* of I:

(2.1) $$rank(I) = \dim_{C(x)}(C(x)[\xi]/C(x)[\xi] \cdot \mathrm{in}_{(0,e)}(I)).$$

Here $C(x) = C(x_1, \ldots, x_n)$. The holonomic rank equals the dimension of the C-vector space of holomorphic solutions to I at any point outside the singular locus.

Theorem 2.1. *Let I be a holonomic D-ideal and $w \in R^n$. Then $\mathrm{in}_{(-w,w)}(I)$ is holonomic and*

(2.2) $$rank(\mathrm{in}_{(-w,w)}(I)) \leq rank(I).$$

Our proof of Theorem 2.1 is based on a walk in the *Gröbner fan* $GF(I)$ as defined in [1]. This fan decomposes the closed half space $\{u + v \geq 0\}$ of R^{2n} into finitely many convex polyhedral cones, one for each initial monomial ideal $\mathrm{in}_{(u,v)}(I) \subset C[x, \xi]$.

Let $\mathfrak{D}$ be the sheaf of algebraic differential operators on C^n. A holonomic D-ideal I is called

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regular holonomic if the $\mathfrak{D}$-module $\mathfrak{D}/\mathfrak{D}I$ is regular holonomic in the sense of [9] or [2, Def. 11.3 (ii), p. 302].

Theorem 2.2. *Let I be a regular holonomic D-ideal and w any weight vector. Then*

(2.3) $rank(I) = rank(\mathrm{in}_{(-w,w)}(I))$.

For the special case $w = e$ and assuming $\lambda_\beta - \lambda_{\beta'} \notin \boldsymbol{Z}$ as in Theorem 4.2 below, the identity (2.3) is a consequence of [8, Theorem 1.1]. Our proof of Theorem 2.2 in general is independent of [8] and more elementary. It is based on Theorem 2.1 and the construction of the canonical series solutions to I in the next section.

3. Series solutions with logarithms. Let I be a regular holonomic D-ideal and $w \in \boldsymbol{R}^n$ generic. Then $J := \mathrm{in}_{(-w,w)}(I)$ is torus-fixed. The artinian ideal $\tilde{J} \subset \boldsymbol{C}[\theta]$ is called the *indicial ideal* of I with respect to w. Let $V(J) = \{\beta_1, \ldots, \beta_p\} \subset \boldsymbol{C}^n$ denote the zero set of $\tilde{J}$. This set is finite since $\tilde{J}$ is artinian. The vectors β_i are called the *exponents* of I with respect to w.

The *Gröbner cone* of I containing w is the open convex polyhedral cone

$$C_w(I) = \{w' \in \boldsymbol{R}^n : \mathrm{in}_{(-w',w')}(I) = J\}.$$

This is a maximal cone in the restriction of the Gröbner fan $GF(I)$ to $\{u + v = 0\}$. Its polar dual $C_w(I)^*$ is closed and strongly convex. It consists of all $\nu \in \boldsymbol{R}^n$ such that $\mathrm{in}_{(-w',w')}(I) = J$ implies $\nu \cdot w' \geq 0$. Let $\boldsymbol{C}[[C_w(I)^*_{\boldsymbol{Z}}]]$ be the ring of formal power series $f = \sum_u c_u x^u$ where $c_u \in \boldsymbol{C}$ and $u \in C_w(I)^* \cap \boldsymbol{Z}^n$. Note that the *initial form* $\mathrm{in}_w(f) := \sum_{u:w\cdot u \text{ minimal}} c_u x^u$ is well-defined, since $u \cdot w > 0$ for all $u \in C_w(I)^* \backslash \{0\}$.

Theorem 3.1. *There are $rank(I)$ many $\boldsymbol{C}$-linearly independent series in the ring*

$$R = \boldsymbol{C}[[C_w(I)^*_{\boldsymbol{Z}}]][x^{\beta_1}, \ldots, x^{\beta_p}, \log(x_1), \ldots, \log(x_n)]$$

which are annihilated by I and have a common domain of convergence in $\boldsymbol{C}^n$.

The weight vector $w \in \boldsymbol{R}^n$ defines a partial order on the monomial basis of R:

(3.1) $x^a \log(x)^b \leq x^c \log(x)^d :\Leftrightarrow Re(w\cdot a) \leq Re(w\cdot c)$.

Here $Re(w\cdot a)$ denotes the real part of the complex number $w \cdot a$. Let $g \in R$. The *initial form* $\mathrm{in}_w(g)$ is the finite sum of terms $c_{ab}x^a \log(x)^b$ in g minimal under (3.1).

Lemma 3.2. *If g is annihilated by I then $\mathrm{in}_w(g)$ is annihilated by $J = \mathrm{in}_{(-w,w)}(I)$.*

Let $\prec_w$ be the refinement of the partial order (3.1) by the lexicographic order $\prec$ on the exponents $(a, b) \in \boldsymbol{C}^n \oplus \boldsymbol{N}^n \simeq \boldsymbol{R}^{2n} \oplus \boldsymbol{N}^n$. Each $g \in R$ has a unique *initial monomial* $\mathrm{in}_{\prec_w}(g) = x^a \log(x)^b$. Consider the following set of *starting monomials*:

$Start_{\prec_w}(I) := \{\mathrm{in}_{\prec_w}(g) : g \in R\backslash\{0\}$ is annihilated by $I\}$.

We next construct the $\boldsymbol{C}$-basis of *canonical series solutions* to I with respect to $\prec_w$.

Theorem 3.3. *The cardinality of $Start_{\prec_w}(I)$ equals $rank(I)$. For each $x^a \log(x)^b \in Start_{\prec_w}(I)$ there is a unique element $g \in R\backslash\{0\}$ with the following properties:*

(a) g is annihilated by I;

(b) $\mathrm{in}_{\prec_w}(g) = x^a \log(x)^b$;

(c) No starting monomial other than $x^a \log(x)^b$ appears in the expansion of g.

4. Algorithmic Frobenius method. If a torus-fixed D-ideal J is holonomic, then $\tilde{J}$ is artinian, and in this case,

(4.1) $rank(J) = rank(D\cdot\tilde{J}) = dim_{\mathrm{C}}(\boldsymbol{C}[\theta]/\tilde{J})$.

Solutions in R to J are determined from the primary decomposition

$$\tilde{J} = \bigcap_{\beta \in V(J)} J_\beta(\theta - \beta).$$

Here J_β is an artinian ideal primary to the maximal ideal $\langle \theta_1, \ldots, \theta_n \rangle$ in $\boldsymbol{C}[\theta]$. A $\boldsymbol{C}$-basis for its orthogonal complement $J_\beta^\perp$ is derived from the term order $\prec$ by *Gröbner duality* as in [10], [11].

Proposition 4.1. *The canonical solutions to J are $x^\beta \cdot p(log(x_1), \ldots, log(x_n))$ where $\beta \in V(J)$ and p is in the $\boldsymbol{C}$-basis of $J_\beta^\perp$ dual to the reduced $\prec$-Gröbner basis of J_β.*

Let I be a regular holonomic D-ideal and $w \in \boldsymbol{R}^n$ generic. If $g \in R$ is a canonical solution of I then $\mathrm{in}_{(-w,w)}(g)$ is a canonical solution of $J = \mathrm{in}_{(-w,w)}(I)$ and hence computed by Proposition 4.1. Our goal is to reconstruct g from $\mathrm{in}_{(-w,w)}(g)$. The following result is a consequence of our algorithmic Frobenius method [15] and a generalization of the method in [7]. The hypothesis $\lambda_\beta - \lambda_{\beta'} \notin \boldsymbol{Z}$ in Theorem 4.2 is still unsatisfactory. We hope to be able to remove it in the final version of [15].

Let J be the torus fixed ideal in $D\langle t, \partial_t \rangle$ generated by $I_0 = \mathrm{in}_{(-w,w)}(I)$ and $\theta_t - \sum_{i=1}^n w_i\theta_i$. Let $b_0(\theta_t)$ be the generator of $\tilde{J} \cap \boldsymbol{C}[\theta_t]$. Consider the primary decomposition $\tilde{J} = \cap_{\beta \in V(I_0)} J_{(\beta,\lambda_\beta)}(\theta - (\beta, \lambda_\beta))$ where $\lambda_\beta = \sum_{i=1}^n w_i\beta_i$. Since w is generic, we may assume that there exist one-to-one correspondences between the points of $V(J)$, the points of $V(I_0)$, and the roots λ_β of $b_0(s) = 0$.

We identify these points. Consider the $\boldsymbol{C}$-vector subspace $J_\beta^* = \{p(\partial_\mu, \partial_\varepsilon) \mid p \in J_{(\beta,\lambda_\beta)}^\perp\}$ of the Weyl algebra over $\mu_1, \ldots, \mu_n, \varepsilon$. We call it the space of *Frobenius jets* with respect to the exponent β. We extend the term order $\prec$ arbitrarily to include the new variable θ_t.

Theorem 4.2. *Assume that the b-function* $b_0(s)$ *is factored as*

$$b_0(s) = \prod_{\beta \in V(I_0)} (s - \lambda_\beta)^{\nu_\beta}, \text{ with } \lambda_\beta - \lambda_{\beta'} \notin \boldsymbol{Z} \text{ for } \beta \neq \beta'.$$

Let $J_{\beta,\prec}^*$ *be the* $\boldsymbol{C}$*-basis of the Frobenius jets* J_β^* *which is dual to the reduced* $\prec$*-Gröbner basis of the primary ideal* $J_{(\beta,\lambda_\beta)}$*. For each exponent* $\beta \in V(I_0)$ *one can construct a series* $g_\beta \in \boldsymbol{C}(\mu, \varepsilon)[[C_w(I)_{\boldsymbol{Z}}^*]][[t]]$ *such that the collection of derived series*

$$\lim_{t \to 1} \lim_{\mu,\varepsilon \to 0} x^\beta p(x^\mu t^\varepsilon g_\beta(\mu, \varepsilon; x, t)), \quad \text{for all } \beta \in V(I_0) \text{ and } p \in J_{\beta,\prec}^*,$$

equals the basis of canonical series solutions to I *with respect to* $\prec_w$.

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