On compact conformally flat Einstein-Weyl manifolds

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1. Introduction. Let M be an n-dimensional manifold with a conformal class C. A *conformal connection* on M is an affine connection D preserving the conformal class C, that is, for any Riemannian metric $g \in C$, there exists a 1-form ω_g such that $Dg = \omega_g \otimes g$. We also assume that D is torsion-free. The triple (M, C, D) is called a *Weyl manifold* and D is called a *Weyl structure* on (M, C). A manifold admits an *Einstein-Weyl structure* if there is a Weyl structure for which the symmetric part of the Ricci curvature of the conformal connection is proportional to a metric in C. The Einstein-Weyl equation on the affine connection, which needs an auxilary metric in a given conformal class, is a conformally invariant nonlinear partial differential equation. If (M, g) is an Einstein manifold, then the Levi-Civita connection ∇_g defines an Einstein-Weyl structure of the conformal class $[g]$. Thus the notion of the Einstein-Weyl structure is a generalization of an Einstein metric to conformal structures.

Classically, it is well-known that a conformally flat Einstein manifold must to be a constant curvature manifold. In this paper, as an analogue to this result, we will give classification of closed conformally flat Einstein-Weyl manifolds.

2. Preliminaries. Let (M, C, D) be a Weyl manifold. We assume $n = \dim M \geq 3$. Let Ric^D denote the Ricci curvature of D. In general, Ricci curvature of conformal connection is not symmetric, so we denote by $\mathrm{Sym}(\mathrm{Ric}^D)$ its symmetric part. The scalar curvature R_g^D of D with respect to $g \in C$ is defined by

$$R_g^D = \mathrm{tr}_g \mathrm{Ric}^D. \tag{2.1}$$

A Weyl manifold (M, C, D) is said to be an *Einstein-Weyl manifold* if the symmetric part of the Ricci curvature Ric^D is proportional to the metric g in C. Therefore the *Einstein-Weyl equation* is

$$\mathrm{Sym}(\mathrm{Ric}^D) = \frac{R_g^D}{n} g. \tag{2.2}$$

Note that $R_g^D\, g$ is conformally invariant quantity. In terms of the Ricci curvature and the scalar curvature of the metric $g \in C$, the Einstein-Weyl equation can be written by

$$\mathrm{Ric}_g + \frac{n-2}{4}\left\{ \mathcal{L}_{\omega_g^\#} g + \frac{2}{n}(\delta_g \omega_g) g + \omega_g \otimes \omega_g - \frac{|\omega_g|^2}{n} g \right\} = \frac{R_g}{n} g. \tag{2.3}$$

where $\mathcal{L}$ is the Lie derivative, δ_g is the codifferential of g, and the vector field $\omega_g^\#$ is defined as $\omega_g(X) = g(X, \omega_g^\#)$ for all vector fields X.

We prepare some known facts concerning geometry of Weyl manifolds, which we will use in this paper.

Theorem 2.1 (Gauduchon) ([2]). *Let (M, C, D) be a closed Weyl manifold. Then up to homothety, there exists a unique Riemannian metric g in the conformal class C such that the corresponding 1-form ω_g is co-closed: $\delta_g \omega_g = 0$.*

The metric $g \in C$ is called the *Gauduchon metric* if it is up to homothety the unique metric which satisfies $\delta_g \omega_g = 0$.

Corollary 2.2. *Let (M, C, D) be a closed Einstein-Weyl n-manifold, and $g \in C$ the Gauduchon metric. Then $\omega_g^\#$ is a Killing vector field on (M, g), and Einstein-Weyl equation can be written in the following form:*

$$\mathrm{Ric}_g + \frac{n-2}{4}\left(\omega_g \otimes \omega_g - \frac{|\omega_g|^2}{n} \right) = \frac{R_g}{n} g. \tag{2.4}$$

Theorem 2.3 ([4]). *Let (M, C, D) be a connected closed Einstein-Weyl manifold, and $g \in C$ the Gauduchon metric. If the scalar curvature R_g^D of D with respect to g is non-positive but not identically zero, then (M, g) is Einstein.*

Theorem 2.4 ([4]). *Let (M, C, D) be a closed Einstein-Weyl manifold and $g \in C$ the Gauduchon metric. If $R_g^D > 0$, then the fundamental group $\pi_1(M)$ of M is finite.*

Theorem 2.5 ([4]). *Let (M, C, D) be a closed connected non-trivial Einstein-Weyl manifold with $R_g^D = 0$. Then $b_1(M) = 1$.*

Lemma 2.6. *Let (M, C, D) be a connected* *closed Einstein-Weyl manifold, and* $g \in C$ *the Gauduchon metric. Then* $R_g - \frac{n+2}{4}|\omega_g|^2 =$ *const.*

Proof. A direct calculation with the second Bianchi identity: $\delta_g \mathrm{Ric}_g + \frac{1}{2} R_g = 0$, and using the Gauduchon metric.

3. Main result. Theorem 3.1. *Let* (M, C, D) *be an* n*-dimensional closed conformally flat Einstein-Weyl manifold,* $n \geq 3$. *Then* (M, C, D) *is either*

(*1*) *A trivial Einstein-Weyl structure induced by a constant curvature metric,*

or

(*2*) *The type* $S^1 \times S^{n-1}$.

We prepare the following Lemma:

Lemma 3.2. *Let* (M, C, D) *be a closed connected conformally flat Einstein-Weyl manifold, and* $g \in C$ *the Gauduchon metric. Then the scalar curvature* R_g *of* g, R_g^D *of* D *with respect to* g, *and the norm* $|\omega_g|$ *of the corresponding 1-form are all constants.*

Proof. Because (M, g) is conformally flat, we have

$$(3.1)\quad \begin{aligned} &\nabla_Y \mathrm{Ric}_g(X, Z) - \nabla_Z \mathrm{Ric}_g(X, Y) \\ &- \frac{1}{2(n-1)}\{\nabla_Y R_g g(X, Z) - \nabla_Z R_g g(X, Y)\} = 0. \end{aligned}$$

On the other hand, g is the Gauduchon metric, from the Einstein-Weyl equation, Ricci curvature Ric_g of g is written by

$$(3.2)\quad \mathrm{Ric}_g = \frac{R_g}{n} g - \frac{n-2}{4}\left(\omega_g \otimes \omega_g - \frac{|\omega_g|^2}{n} g\right),$$

so we get

$$(3.3)\quad \begin{aligned} &\frac{1}{n-1}(\nabla_Y R_g g(X, Z) - \nabla_Z R_g(X, Y)) \\ &- \frac{n}{2}\Big\{\nabla_Y \omega_g(X)\omega_g(Z) + \nabla_Y \omega_g(Z)\omega_g(X) \\ &- \nabla_Z \omega_g(X)\omega_g(Y) \\ &- \nabla_Z \omega_g(Y)\omega_g(X) - \frac{1}{n}\nabla_Y |\omega_g|^2 g(X, Z) \\ &+ \frac{1}{n}\nabla_Z |\omega_g|^2 g(X, Y)\Big\} \\ &= 0. \end{aligned}$$

By taking a trace,

$$(3.4)\quad dR_g - \frac{n}{2}\omega_g \cdot \nabla \omega_g - \frac{1}{2} d|\omega_g|^2 = 0.$$

Now $\omega_g^\#$ is a Killing vector field, so

$$(3.5)\quad \mathcal{L}_{\omega_g^\#} g = 0,$$

we get $d(R_g - \frac{n-2}{4}|\omega_g|^2) = 0$. On the other hand, $R_g - \frac{n+2}{4}|\omega_g|^2 = \text{const}$. So $R_g = \text{const.}$, and $|\omega_g| = \text{const}$, and $R_g^D = R_g - \frac{(n-1)(n-2)}{4}|\omega_g|^2$, we get $R_g^D = \text{const}$.

Proof of Theorem 3.1. Note that R_g^D is constant, so we consider three cases.

Firstly R_g^D is negative. In this case, the Gauduchon metric is a conformally flat Einstein metric, so it is a hyperbolic metric.

Nextly R_g^D is positive. In this case, the fundamental group is finite, so the universal covering space is simply connected compact conformally flat. From Kuiper's theorem ([3]), that is conformally diffeomorphic to the standard sphere. Note that

$$(3.6)\quad -\Delta_g \omega_g = 2\mathrm{Ric}_g(\omega_g^\#) = 2(n-1)\omega_g,$$

and

$$(3.7)\quad -\Delta_g \omega_g = \frac{2}{n} R_g^D \omega_g,$$

so we have $\omega_g = 0$. In this case (M, C, D) is the trivial Einstein-Weyl structure induced by the standard sphere.

In the last case, R_g^D is identically zero. If $\omega_g = 0$, then this is a trivial Einstein-Weyl structure induced by the Euclidean metric. We assume $\omega_g \neq 0$. We have then

$$(3.8)\quad \mathrm{Ric}_g = -\frac{n-2}{4}(\omega_g \otimes \omega_g - |\omega_g|^2 g) \geq 0.$$

So note that $b_1(M) = 1$ and from the Cheeger-Gromoll splitting theorem ([1]) for manifolds of non-negative Ricci curvature, the universal covering of M is diffeomorphic to $\boldsymbol{R} \times S^{n-1}$.

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