Delone sets and Riesz basis

By Hideo SUZUKI

Graduate School of Mathematics, Kyushu University

(Communicated by Kiyosi ITÔ, M. J. A., Jan. 12, 1998)

Abstract: In this paper we deal with the density of Delone set and apply it constructing Riesz basis for an Hilbert space.

1. Introduction. A Riesz basis for Hilbert space is easily constructed by exponential maps over a periodic set. This drives us to the question how it is when a periodic set is replaces by Delone set. Construction by exponential functions will certainly work if a Delone set is very close to a periodic set. We are concerned with the problem how a Delone set can be different from the periodic set. In fact, Kadec and Levinson studied such a problem in the case of $L^p[-\pi, \pi]$ (p is a natural number) (see [6] pp. 118-131).

The purpose of the present note is to explore a little further in the cases of $L^2[-\pi, \pi]$ and $H^1[-\pi, \pi]$ (see our main theorem 5.3 and 5.4 below [6]).

2. Delone set and Voronoi cell.

Definition 2.1. An (R, r)-Delone set $\Lambda \subset R^N$ is defined by the next two conditions (see [5] p. 28).

1) Discretness: There exists a positive real number r such that for every $x, y \in \Lambda$, $|x-y| \geq 2r$.
2) Relative density: There is a positive real number R such that every sphere of radius greater than R contains at least one point of Λ in its interior.

Definition 2.2. Let $\Lambda \subset R^N$ be any Delone set. The Voronoi cell at a point $x \in \Lambda$ is the set of points of R^N that lie at least as close to x as to any other point of Λ:

$$V(x) = \{u \in R^N \| x - y| \leq |y-u|,\ y \in \Lambda\}.$$

The Voronoi cell $V(x)$ is then the smallest convex region about x (see [5] p. 42).

If Λ is a lattice, the Voronoi cells are congruent.

Here we deal with a Delone set Λ including $0 : 0 \in \Lambda$.

3. The density of Delone set. We introduce the notion of the density for the (R, r)-Delone set. The density Δ of Delone set Λ centered at x is defined by

$$(1)\qquad \Delta_x(\Lambda) = \lim_{s\to\infty} \frac{\#\{\Lambda \cap B_x(s)\}}{m(B_x(s))}$$

(m is the Lebesgue measure).

If (1) is well-defined, we say that Λ has the density $\Delta_x(\Lambda)$ at x.

Here, we should notice that $\Delta_x(\Lambda)$ is actually independent of $x \in R^N$.

Lemma 3.1.

$$(2)\qquad \Delta_x(\Lambda) = \Delta_0(\Lambda),$$

for all x.

Proof. Let $\Delta_x(\Lambda)$ be defined for a fixed $x \in R^N$.

Take $s > 0$ such that $s > |x|$. Then

$$B_x(s - |x|) \subset B_0(s) \subset B_x(s + |x|).$$

Here $|x| = \sqrt{x_1^{\ 2} + x_2^{\ 2} + \cdots + x_N^{\ 2}}$ for $x = (x_1, x_2, \ldots, x_N)$.

Therefore

$$B_x(s-|x|) \cap \Lambda \subset B_0(s) \cap \Lambda \subset B_x(s+|x|) \cap \Lambda.$$

We obtain

$$\frac{\#\{B_x(s-|x|) \cap \Lambda\}}{m(B_0(s))} \leq \frac{\#\{B_0(s) \cap \Lambda\}}{m(B_0(s))} \leq \frac{\#\{B_x(s+|x|) \cap \Lambda\}}{m(B_0(s))}.$$

$$\frac{\#\{B_x(s-|x|) \cap \Lambda\}}{m(B_0(s-|x|))}\left\{\frac{s-|x|}{s}\right\}^N \leq \frac{\#\{B_0(s) \cap \Lambda\}}{m(B_0(s))} \leq \left(\frac{\#\{B_x(s+|x|) \cap \Lambda\}}{m(B_0(s+|x|))}\right)\left\{\frac{s+|x|}{s}\right\}^N$$

($\#$ is the number of elements).

We have (2) when $s \to \infty$. □

Corollary 3.2.

$$\Delta_x(\Lambda) = \Delta_y(\Lambda),$$

for $x \neq y$.

We now define the density of Λ by

$$(3)\qquad \Delta(\Lambda) = \lim_{s\to\infty} \frac{\#\{\Lambda \cap B_0(s)\}}{m(B_0(s))}$$

and call $\Delta(\Lambda)$ the density of Λ.

For a (R, r)-Delone set,
(minimal volume of Voronoi cells) $\leq \frac{1}{\Delta(\Lambda)} \leq$ (maximal volume of Voronoi cells).

Thus,

$$O\left\{\frac{1}{R^N}\right\} \leq \Delta(\Lambda) \leq O\left\{\frac{1}{r^N}\right\}.$$

If Λ is a lattice, Landau showed the following results [1], [2], and [3].

Theorem 3.3.

$$\Delta(Z^N) = 1.$$

Theorem 3.4. Let A be a regular $N \times N$ real matrix. then

$$\Delta(A(Z^N)) = |\det A|^{-1}. \tag{4}$$

That is, $\frac{1}{\Delta(\Lambda)}$ is the volume of each Voronoi cell in Λ.

4. Special Delone set. Definition 4.1. Let Λ be a set including 0. We say Λ is an L-special set ($L < 1/4$) if there exists a regular matrix A such that

a) $\#(D_L(n) \cap A^{-1}(\Lambda)) = 1$ for any $n \in Z^N$.

b) $A^{-1}(\Lambda) \subset \cup_{n \in Z^N} D_L(n)$.

Here $D_L(n) = \{x \in R^N \mid |x_j - n_j| < L;\ 1 \leq j \leq N\}$ for $n = (n_1, \ldots, n_N)$.

A is called a lattice matrix and $\Lambda' = A(Z^N)$ the periodic lattice associated with Λ.

Lemma 4.2. If Λ is an L-special set, it is a Delone set. We freshly call Λ L-special Delone set. Λ' is not unique, but we see.

Lemma 4.3. Let Λ be an L-special Delone set. Then,

$$\Delta(\Lambda) = \Delta(\Lambda') = |\det A|^{-1}. \tag{5}$$

Λ' is the periodic lattice associated with Λ.

Proof. (5) is a consequence of Theorem 3.4 and the next relation:

$$\#(\Lambda \cap B_0(s - 2R)) \leq \#(\Lambda' \cap B_0(s)) \leq \#(\Lambda \cap B_0(s + 2R))$$

valid for all $s > 2R$.

Then, we have (5). □

5. Delone set and Riesz basis. Recall that a basis $\{r_n\}$ of a Hilbert space X is a Riesz basis if there is a bounded invertible operator T and an orthonormal basis $\{b_n\}$ in X such that $r_n = Tb_n$ for all n ([6] p. 31).

Theorem 5.1 (Kadec's 1/4 theorem). Let $\{\lambda_n\}$ satisfy $\sup|\lambda_n - n| < L < 1/4$. Then $\{\exp i\lambda_n t\}$ is a Riesz basis for $L^2[-\pi, \pi]$ (see [6] p. 42).

Lemma 5.2. Let $\{e_n\}$ be an orthonormal basis for a Hilbert space H. Suppose $\{f_n\} \subset H$ be "close enough" to $\{e_n\}$ in the sense that

$$\|\sum c_k(e_k - f_k)\|_2 \leq \mu\sqrt{\sum |c_k|^2}, \tag{6}$$

for some constant μ; $0 \leq \mu < 1$, and arbitrary scalars $\{c_n\}$ ($\|\cdot\|_2$ is the $L^2[-\pi, \pi]$-norm).

Then $\{f_n\}$ is a Riesz basis for $L^2[-\pi, \pi]$ (see [6] p. 40).

Our first main result reads as follows.

Theorem 5.3. Let Λ be an L-special Delone set associated with a periodic lattice $\Lambda' = A(Z^N)$ on R^N.

If $\Lambda(\Delta)$ satisfy a) or b):

(7) a) $\Delta(\Lambda) \leq \frac{1}{(2^N - 1)^2}$

b) $\Delta(\Lambda) > \frac{1}{(2^N - 1)^2}$ and

$$L < \frac{1}{4} - \frac{1}{\pi}\sin^{-1}\frac{2 - \sqrt[N]{1 + \Delta(\Lambda)^{-\frac{1}{2}}}}{\sqrt{2}},$$

then $\{\exp(i\lambda \cdot x)\}_{\lambda \in \Lambda}$ forms a Riesz basis for $L^2(W_A(0))$. Here $W_A(0) = (2\pi)^T A^{-1}V(0))$ for the Voronoi cell $V(0)$ at $0 \in Z^N$, and $\lambda \cdot x$ is the inner product of λ, $x \in R^N$.

Proof. We denote the unique element $\lambda \in \{A(V(0)) + Ak\} \cap \Lambda$ by λ_k, $k \in Z^N$.

Since $\{\exp(i\lambda' \cdot x)\}_{\lambda' \in \Lambda'}$ forms an orthonormal basis for $L^2(W_A(0))$, we have to show by Lemma 5.2 that

$$\left\|\sum_{k \in Z^N} c_k(\exp(i\lambda_k \cdot x) - \exp(iAk \cdot x))\right\|_{L^2(W_A(0))}$$

< 1 whenever $\sum |c_k|^2 \leq 1$.

Since $A^{-1}\lambda_k \in D_L(k)$, we set by the triangle inequality and Theorem 5.1,

$$\left\|\sum_{k \in Z^N} c_k(\exp(i\lambda_k \cdot x) - \exp(iAk \cdot x))\right\|_{L^2(W_A(0))} \tag{8}$$
$$\leq |\det A|^{-\frac{1}{2}}\left\|\sum_{k \in Z^N} c_k(\exp(iA^{-1}\lambda_k \cdot y) - \exp(ik \cdot y))\right\|_{L^2(V(0))}$$
$$\leq |\det A|^{-\frac{1}{2}}\{(2 - \cos\pi L + \sin\pi L)^N - 1\}.$$

(7) implies that the right hand of (8) is smaller than 1. □

Our second main result reads as follows:

Theorem 5.4. Let Λ be an L-special Delone set associated with Z. If

$$2(1 - \cos\pi L + \sin\pi L)^2 + 8L^2 < 1, \tag{9}$$

then $\left\{\frac{\exp iat}{\sqrt{a^2 + 1}}\right\}_{a \in \Lambda}$ is a Riesz basis for $H^1[-\pi, \pi]$.

Proof. As $\left\{\frac{\exp ikx}{\sqrt{k^2 + 1}}\right\}_{k=-\infty}^{\infty}$ forms an orthonormal basis for $H^1[-\pi, \pi]$, we have to show by Lemma 5.2 that

$$\left\| \Sigma\, c_k \left(\frac{\exp ia_k x}{\sqrt{a_k^{\,2}+1}} - \frac{\exp ikx}{\sqrt{k^2+1}} \right) \right\|_{H^1}^2 < 1$$

whenever $\Sigma |c_k|^2 \leq 1$ ($\|\cdot\|_{H^1}$ is the $H^1[-\pi, \pi]$-norm).

$$\left\| \Sigma\, c_k \left\{ \frac{\exp ia_k x}{\sqrt{a_k^{\,2}+1}} - \frac{\exp ikx}{\sqrt{k^2+1}} \right\} \right\|_{H^1}^2$$

$$\leq 2 \left\| \Sigma \frac{c_k}{\sqrt{a_k^{\,2}+1}} (\exp ia_k x - \exp ikx) \right\|_2^2$$

$$+ 2 \left\| \Sigma\, c_k \left\{ \frac{1}{\sqrt{a_k^{\,2}+1}} - \frac{1}{\sqrt{k^2+1}} \right\} \exp ikx \right\|_2^2$$

$$+ 2 \left\| \Sigma \left\{ \frac{c_k a_k}{\sqrt{a_k^{\,2}+1}} \right\} (\exp ia_k x - \exp ikx) \right\|_2^2$$

$$+ 4 \left\| \Sigma \left\{ \frac{c_k (a_k - k)}{\sqrt{a_k^{\,2}+1}} \exp ikx \right\} \right\|_2^2$$

$$+ 4 \left\| \Sigma c_k k \left\{ \frac{1}{\sqrt{a_k^{\,2}+1}} - \frac{1}{\sqrt{k^2+1}} \right\} \exp ikx \right\|_2^2.$$

Recall that if $\sup |a_k - k| \leq L < 1/4$,

$$\|\Sigma c_k (\exp ia_k x - \exp ikx)\|_2^2 < (1 - \cos \pi L + \sin \pi L)^2 < 1$$

for $\Sigma |c_k|^2 < 1$. Note also

$$\Sigma \left| \frac{c_k}{\sqrt{a_k^{\,2}+1}} \right|^2 \leq \Sigma |c_k|^2 < 1,$$

$$\Sigma \left| \frac{c_k a_k}{\sqrt{a_k^{\,2}+1}} \right|^2 \leq \Sigma |c_k|^2 < 1,$$

(10)

$$\left\| \Sigma \frac{c_k}{\sqrt{a_k^{\,2}+1}} (\exp ia_k x - \exp ikx) \right\|_2^2$$

$$+ \left\| \Sigma \frac{c_k a_k}{\sqrt{a_k^{\,2}+1}} (\exp ia_k x - \exp ikx) \right\|_2^2$$

$$\leq (1 - \cos \pi L + \sin \pi L)^2 \Sigma \frac{|c_k|^2}{a_k^{\,2}+1}$$

$$+ (1 - \cos \pi L + \sin \pi L)^2 \Sigma \frac{a_k^{\,2} |c_k|^2}{a_k^{\,2}+1}$$

$$\leq (1 - \cos \pi L + \sin \pi L)^2 \Sigma |c_k|^2$$

$$< (1 - \cos \pi L + \sin \pi L)^2.$$

On the other hand,

(11)

$$\left\| \Sigma\, c_k \left\{ \frac{1}{\sqrt{a_k^{\,2}+1}} - \frac{1}{\sqrt{k^2+1}} \right\} \exp ikx \right\|_2^2$$

$$+ 2 \left\| \Sigma \frac{c_k (a_k - k)}{\sqrt{a_k^{\,2}+1}} \exp ikx \right\|_2^2$$

$$+ 2 \left\| \Sigma\, c_k k \left\{ \frac{1}{\sqrt{a_k^{\,2}+1}} - \frac{1}{\sqrt{k^2+1}} \right\} \exp ikx \right\|_2^2$$

$$\leq \Sigma |c_k|^2 \left\{ \frac{\sqrt{k^2+1} - \sqrt{a_k^{\,2}+1}}{\sqrt{a_k^{\,2}+1}\sqrt{k^2+1}} \right\}^2$$

$$+ 2\Sigma |k c_k|^2 \left\{ \frac{\sqrt{k^2+1} - \sqrt{a_k^{\,2}+1}}{\sqrt{a_k^{\,2}+1}\sqrt{k^2+1}} \right\}^2$$

$$+ 2\Sigma |c_k|^2 \left\{ \frac{a_k - k}{\sqrt{k^2+1}} \right\}^2$$

$$\leq \Sigma |c_k|^2 \left\{ \frac{(2k^2+1)(\sqrt{k^2+1} - \sqrt{a_k^{\,2}+1})}{\sqrt{(k^2+1)(a_k^{\,2}+1)}} \right\}^2$$

$$+ \Sigma |c_k|^2 \left\{ \frac{a_k - k}{\sqrt{k^2+1}} \right\}^2$$

$$\leq \Sigma |c_k|^2 \left\{ \frac{4k^2+3}{(k^2+1)(a_k^{\,2}+1)} (k - a_k)^2 \right\}$$

$$\leq 4\Sigma |c_k|^2 \left\{ \frac{(a_k - k)^2}{a_k^{\,2}+1} \right\}$$

$$\leq 4L^2 \Sigma \frac{|c_k|^2}{a_k^{\,2}+1}$$

$$\leq 4L^2.$$

By using (9), (10), and (11),

$$\left\| \Sigma\, c_k \left(\frac{\exp ia_k x}{\sqrt{a_k^{\,2}+1}} - \frac{\exp inx}{\sqrt{k^2+1}} \right) \right\|_{H^1}^2$$

$$< 2(1 - \cos \pi L + \sin \pi L)^2 + 8L^2$$

$$< 1.$$

□

References

[1] E. Landau: Einfürung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale. Teubner 2nd Ed. (1927).

[2] E. Landau: Vorlesungen über Zahlentheorie II. Hirzel (1927).

[3] E. Landau: Zur analytischen Zahlentheorie der definiten quadratischen Formen. S.-B. Preuss. Akad. Wiss, pp. 458–476 (1915).

[4] John J. Benedetto: Irregular Sampling and Frames. (eds. Wavelets-A Tutorial in Theory and Applications C. K. Chui). Academic Press Inc., pp. 445–507 (1992).

[5] M. Senechal: Quasicrystals and Geometry. Cambridge University Press, Cambridge (1995).

[6] R. M. Young: An Introduction to Nonharmonic Fourier Series. Academic Press Inc., New York (1980).