Odd Dimensional Tori and Contact Structure

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1. We investigate the problem whether or not the odd dimensional torus $T^{2n+1} (2n+1>3)$ admits a contact structure. This problem was posed on the five dimensional torus T^5 by D.E. Blair in his lecture note [1] and he showed in [1] that no torus T^{2n+1} can carry a regular contact structure.

We remark that T^3 carries a contact structure $\eta_o = \cos z dx + \sin z dy$ which is neither regular nor K-contact.

In this note we will exhibit two theorems related to the problem, of which the first one is concerned with non-existence of a K-contact structure and the second with a certain kind of one-forms not yielding a contact structure on T^5. More precisely, the theorems which we will give are the following

Theorem 1. No torus T^{2n+1} can carry a K-contact structure

and

Theorem 2. Denote by (x, y, z, u, v) the canonical coordinate in T^5. Then any one-form η of the form

$$\eta = \cos z\, dx + \sin z\, dy + f\, du + h dv$$

can not be a contact structure, provided that the functions $f, h \in C^\infty(T^5)$ satisfy either

$$\frac{\partial f}{\partial y} = \frac{\partial h}{\partial x} = 0$$

or

$$\frac{\partial f}{\partial x} = \frac{\partial h}{\partial y} = 0.$$

We briefly recall the notion of contact structure by following [1].

A $(2n+1)$-dimensional manifold M is called a contact manifold when M admits a one-form η, called a contact form or a contact structure, for which the $(2n+1)$-form $\eta \wedge (d\eta)^n$ gives a volume form on M.

We call a contact structure η regular when the characteristic vector field ξ is regular.

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When a manifold M admits a contact structure η and also a metric g and a (1,1)-tensor ϕ for which the following are satisfied, (η, ϕ, g) is called a contact metric structure;

$$\phi(\phi(X)) = -X + \eta(X)\xi,$$
$$g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y),$$
$$d\eta(X, Y) = 2g(X, \phi(Y)).$$

A contact metric structure (η, ϕ, g) is called K-contact when ξ is a Killing field with respect to g.

2. K-contact structures. Theorem 1 is an immediate consequence of the following theorem, since T^{2n+1} satisfies the cohomology condition in it.

Theorem 3. Let M be a $(2n+1)$-dimensional compact connected manifold. If the cohomology ring $H^*(M, \boldsymbol{R})$ satisfies

$$\wedge^{2n+1}(H^1(M, \boldsymbol{R})) = H^{2n+1}(M, \boldsymbol{R}),$$

then M can not admit a K-contact structure.

Theorem 3 is shown by applying Tachibana's theorem ([1] and [2]). In fact we have

Proof. Tachibana's theorem asserts that every harmonic one-form ω on a compact K-contact manifold satisfies $\omega(\xi) = 0$.

Suppose M admits a K-contact structure (η, ξ, ϕ, g). Then, from the cohomological assumption the $(2n+1)$-cohomology class $[dvol]$ represented by the volume form $dvol$ is given by a linear combination of $(2n+1)$-cohomology classes $[\Omega_j]$ which are represented by $(2n+1)$-exterior products of harmonic one-forms. Let $\Omega_j = \omega_1 \wedge \cdots \wedge \omega_{2n+1}$ be such a $(2n+1)$-form.

With respect to an orthonormal basis $\{e_1 = \xi, e_2, e_3, \cdots, e_{2n+1}\}$ the value $(\omega_1 \wedge \cdots \wedge \omega_{2n+1})(e_1, e_2, \cdots, e_{2n+1}) = \det(\omega_i(e_j))$ vanishes because $\omega_i(e_1) = \omega_i(\xi) = 0$ for all i. This shows that $[dvol] = 0$, yielding a contradiction. So we have Theorem 3.

Remark. Examples of manifold satisfying the cohomological condition in Theorem 3 other that T^{2n+1} are given, for instance, by the product manifolds $M = \Sigma_{g_1} \times \cdots \times \Sigma_{g_\ell} \times S^1$, $g_i > 1$

for each $1 \leq i \leq \ell$, where Σ_g is a compact Riemann surface of genus g.

3. The T^5 case. We consider on the 5-dimensional torus T^5 a one-form $\eta = \eta_o + \alpha$, where $\eta_o = \cos z dx + \sin z dy$ and $\alpha = f du + h dv$; $f, h \in C^\infty(T^5)$. Here (x, y, z, u, v) is the canonical coordinate in T^5.

Then by an easy calculation we have

$$\eta \wedge d\eta \wedge d\eta = 2\eta_o \wedge d\eta_o \wedge d\alpha + 3 d\eta_o \wedge \alpha \wedge d\alpha - d\theta$$

where θ is a 4-form defined by $\theta = \eta_o \wedge \alpha \wedge d\alpha$.

Lemma. The 5-form $\eta \wedge d\eta \wedge d\eta + d\theta$ is represented as

$$\begin{aligned}\eta \wedge d\eta \wedge d\eta + d\theta &= -4(h_u - f_v) dx \wedge dy \wedge dz \wedge du \wedge dv \\ &\quad + 3\{\sin z(f h_y - h f_y) + \cos z(f h_x - h f_x)\} dx \wedge dy \wedge dz \wedge du \wedge dv.\end{aligned}$$

Theorem 2 is derived from this lemma. The proof of Theorem 2 is actually given as follows.

Integrate over T^5 the both hand sides of the above equation. Since the first term of the right hand side is written as an exact form

$$\begin{aligned}(h_u - f_v) dx \wedge dy \wedge dz \wedge du \wedge dv &= -dh \wedge dx \wedge dy \wedge dz \wedge dv \\ &\quad - df \wedge dx \wedge dy \wedge dz \wedge du,\end{aligned}$$

from Stokes' theorem the second term of the left hand side and the first term of the right hand side do not contribute to the integration. We assume now the condition

$$\frac{\partial f}{\partial y} = \frac{\partial h}{\partial x} = 0.$$

Then $\sin z(f h_y - h f_y) = \sin z f h_y$ so that the integration of the second term of the right hand side reduces to

$$\begin{aligned}\int_{T^5} \sin z f h_y dx dy dz du dv &= \int_0^{2\pi} \cdots \int_0^{2\pi} \sin z f(x, z, u, v) dx dz du dv \\ &\quad \times \int_0^{2\pi} h_y dy.\end{aligned}$$

The function $h = h(y, z, u, v)$ is periodic in y and the period is 2π and then the integral $\int_0^{2\pi} h_y dy$ vanishes. From a similar argument we have the vanishing of the integration of the right hand side, from which it follows that the 5-form $\eta \wedge d\eta \wedge d\eta$ can not give a volume form on M.

From another condition on f, h in Theorem 2 we get similarly the same conclusion.

Remarks. (i) It follows from the proof of Theorem 2 that the 5-form $\eta \wedge d\eta \wedge d\eta$ has a zero point on T^5. Then for small perturbation α' of the one-form $\alpha = f du + g dv$ the one-form $\eta = \eta_o + \alpha'$ can not be a contact structure.

(ii) We can relax the statement of Theorem 2 in the following way.

Theorem 4. Any one-form η of the form

$$\eta = c(x, z) dx + s(y, z) dy + f du + h dv$$

can not give a contact structure on T^5, provided the conditions on f, h in Theorem 2 are fulfilled.

Here $\cos z$ and $\sin z$ in Theorem 2 are replaced by functions $c = c(x, z)$ and $s = s(y, z)$. This theorem is proved in a way quite similar to that of Theorem 2.

References

[1] D. Blair: Contact Manifolds in Riemannian Geometry. Lect. Notes in Math., **509**, Springer-Verlag (1976).

[2] S. Tachibana: On harmonic tensors in compact Sasakian spaces. Tôhoku Math. J., **17**, 271–284 (1965).