On the Local Energy Decay of Higher Derivatives of Solutions for the Equations of Motion of Compressible Viscous and Heat-conductive Gases in an Exterior Domain in $\mathbf{R}^3$

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1. Introduction. Let Ω be an exterior domain in $\mathbf{R}^3$ with compact smooth boundary $\partial\Omega$. We consider the following system

$$(1.1)\quad \begin{cases} \rho_t + \gamma \operatorname{div} v = 0 & \text{in } [0, \infty) \times \Omega, \\ v_t - \alpha\Delta v - \beta\nabla(\operatorname{div} v) + \gamma\nabla\rho \\ \qquad + \omega\nabla\theta = 0 & \text{in } [0, \infty) \times \Omega, \\ \theta_t - \kappa\Delta\theta + \omega \operatorname{div} v = 0 & \text{in } [0, \infty) \times \Omega, \\ v|_{\partial\Omega} = 0,\ \theta|_{\partial\Omega} = 0 & \text{on } [0, \infty) \times \partial\Omega \\ (\rho, v, \theta)(0, x) = (\rho_0, v_0, \theta_0)(x) \\ \qquad \text{in } \Omega, \end{cases}$$

where ρ is the density, $v = {}^T(v_1, v_2, v_3)$ the velocity and θ the absolute temperature, α, γ, κ, and ω are positive numbers and β is a nonnegative number. This system is the linearized equation of motion of compressible viscous and heat-conductive gases in an exterior domain in $\mathbf{R}^3$, which was given by Matsumura and Nishida [6] and Ponce [9]. Concerning the nonlinear problem, the unique existence of smooth solutions globally in time near constant state $(\bar\rho_0, 0, \bar\theta_0)$ was studied by Matsumura and Nishida [8]. Deckelnick [2,3] proved the decay estimates for the solutions of nonlinear problem although the decay rate is weaker than that of Cauchy problem given by Matsumura and Nishida [6,7] and Ponce [9]. Our purpose is to get the decay estimates corresponding to Cauchy problem in the case of an exterior domain, which will be discussed in the forthcoming paper [5]. In our strategy, 1st step is to get local energy decay for the solutions of linearized equations (1.1). Kobayashi [4] proved the local energy decay of lower order derivatives of solutions. But since this system (1.1) is hyperbolic-parabolic type and since the regularity of solutions seems to be governed by the hyperbolic part ρ, we shall need to prove the regularity of solutions. Therefore in this paper we discuss a local energy decay estimates for higher order derivatives of solutions for the linearized equations.

Now we shall state the main results. Let $1 < q < \infty$, m be an integer and set

$$X_q^m(\Omega) = \{{}^T U : U \in W_q^{m+1}(\Omega) \times \mathbf{W}_q^m(\Omega) \times W_q^m(\Omega)\},\ X_q(\Omega) = X_q^0(\Omega)$$

where ${}^T U$ means the transposed U, $W_q^m(\Omega) = \{u \in L_q(\Omega) : \|u\|_{m,q,\Omega} = (\sum_{|\alpha|\le m} \int_\Omega |\partial_x^\alpha u|^q dx)^{1/q} < \infty\}$ denotes the usual Sobolev spaces and $\mathbf{W}_q^m(\Omega) = \{W_q^m(\Omega)\}^3$. Define the 5×5 matrix operator A by the relation:

$$A = \begin{pmatrix} 0 & \gamma \operatorname{div} & 0 \\ \gamma\nabla & -\alpha\Delta - \beta\nabla \operatorname{div} & \omega\nabla \\ 0 & \omega \operatorname{div} & -\kappa\Delta \end{pmatrix}$$

with the domain:

$$\mathscr{D}(A) = \{{}^T U = (\rho, v, \theta) \in W_q^1(\Omega) \times \mathbf{W}_q^2(\Omega) \times W_q^2(\Omega) : v|_{\partial\Omega} = 0,\ \theta|_{\partial\Omega} = 0 \text{ on } \partial\Omega\}.$$

Let P be the projection from $\mathscr{D}(A)$ into $\{{}^T(v, \theta) \in \mathbf{W}_q^2(\Omega) \times W_q^2(\Omega);\ v|_{\partial\Omega} = 0,\ \theta|_{\partial\Omega} = 0 \text{ on } \partial\Omega\}$ Then by Kobayashi [4], $-A$ is a closed linear operator in $X_q(\Omega)$ and the resolvent set contain $\Sigma = \{\lambda \in \mathbf{C} : C\mathrm{Re}\lambda + (\mathrm{Im}\lambda)^2 > 0\}$ where C is a constant depending only on α, β, γ, κ, and ω. Moreover, the following properties are valid; There exist positive constants λ_0 and $\delta < \frac{\pi}{2}$ such that

$$(1.2)\quad |\lambda| \|(\lambda + A)^{-1}F\|_{X_q(\Omega)} + \|P(\lambda + A)^{-1}F\|_{2,q,\Omega} \le C(\lambda_0, \delta, m)\|F\|_{X_q(\Omega)}$$

for any $\lambda - \lambda_0 \in \Sigma_\delta = \{\lambda \in C;\ |arg\lambda| \le \pi - \delta\}$ and any $F \in X_q(\Omega)$. This estimates means that $-A$ generates an analytic semigroup e^{-tA} on $X_q(\Omega)$.

Let b be a positive number such that $\partial\Omega \subset B_b = \{x \in \mathbf{R}^3 : |x| < b\}$. Set

$$Y_{q,b}^m(\Omega) = \{U = {}^T(\rho, v, \theta) \in X_q^m(\Omega) : U(x) = 0 \text{ for } x \in \mathbf{R}^3 \backslash B_b,\ \int_{\Omega_b} \rho(x)\,dx = 0\},$$

and $Y_{q,b}(\Omega) = Y_{q,b}^0(\Omega)$ where $\Omega_b = B_b \cap \Omega$. Then

Theorem 1.1. *Let $1 < q < \infty$ and let b_0 be a fixed number such that $B_{b_0} \supset \mathbf{R}^3 \setminus \Omega$. Suppose that $b > b_0$. Then the following estimates are valid; for $M \geq 0$ integers, $\boldsymbol{U} \in \boldsymbol{Y}^1_{q,b}(\Omega)$ and $t \geq 1$*

$$\|\partial_t^M e^{-tA} \boldsymbol{U}\|_{X^1_q(\Omega_b)} + \|\partial_t^M \boldsymbol{P} e^{-tA} \boldsymbol{U}\|_{3,q,\Omega_b} \leq C(q, b, M) t^{-3/2-M} \| \boldsymbol{U} \|_{X^1_q(\Omega_b)}.$$

2. Proof of Theorem 1.1. First we consider the stationary linearized equation with complex parameter λ

(2.1) $(\lambda + \boldsymbol{A})\boldsymbol{U} = \boldsymbol{F}$ in Ω, $\boldsymbol{PU} = 0$ on $\partial\Omega$.

Lemma 2.1. *Let $1 < q < \infty$. Then for $\boldsymbol{F} \in \boldsymbol{X}^1_q(\Omega)$ and $\lambda - \lambda_0 \in \Sigma_\delta$*

$$|\lambda|^{-1/2}\|\boldsymbol{P}(\lambda + \boldsymbol{A})^{-1}\boldsymbol{F}\|_{3,q,\Omega} + |\lambda|^{1/2}\|(1 - \boldsymbol{P})(\lambda + \boldsymbol{A})^{-1}\boldsymbol{F}\|_{2,q,\Omega} \leq C\|\boldsymbol{F}\|_{X^1_q(\Omega)}.$$

Proof. First note that it follows from (1.2) and interpolation theorem that

(2.2) $|\lambda|^{1/2}\|(\lambda + \boldsymbol{A})^{-1}\boldsymbol{F}\|_{1,q,\Omega} \leq C\|\boldsymbol{F}\|_{X_q(\Omega)}$

for $\boldsymbol{F} \in \boldsymbol{X}_q(\Omega)$ and $\lambda - \lambda_0 \in \Sigma_\delta$. Let $\boldsymbol{U} = {}^T(\rho, v, \theta)$, $\boldsymbol{F} = {}^T(f_1, f_2, f_3)$. Applying the elliptic estimates to the system $-\kappa\Delta$ and $-\alpha\Delta - \beta\nabla$ div in (2.1) it follows from (2.2) and (1.2) that

$$\|v\|_{3,q,\Omega} \leq C\{|\lambda|^{1/2}\|\boldsymbol{F}\|_{X_q(\Omega)} + \|\boldsymbol{F}\|_{X^1_q(\Omega)} + |\lambda|^{-1}\|v\|_{3,q,\Omega}\},$$
$$\|\theta\|_{3,q,\Omega} \leq C\{|\lambda|^{1/2}\|\boldsymbol{F}\|_{X_q(\Omega)} + \|\boldsymbol{F}\|_{X^1_q(\Omega)}\},$$
$$\|\rho\|_{2,q,\Omega} \leq C\{|\lambda|^{-1}\{\|f_1\|_{2,q,\Omega} + \|v\|_{3,q,\Omega}\}.$$

Taking λ_0 sufficient large implies this Lemma by these estimates. ■

The following Lemma is concerned with low frequency of resolvent $(\lambda + \boldsymbol{A})^{-1}$ near $\lambda = 0$. Let X and Y be Banach spaces, $\mathscr{B}(X, Y)$ the set of all bounded linear operators from X into Y and $\mathscr{A}(I; X)$ the set of all X-valued holomorphic functions in I. Then

Lemma 2.2. *Let $1 < q < \infty$, bo be a number such that $B_{b_0} \subset \mathbf{R}^3 \setminus \Omega$ and let $b > b_0$. Put $\mathscr{Y} = \mathscr{B}(\boldsymbol{Y}_{q,b}(\Omega); \mathscr{D}(\boldsymbol{A}))$. Then, there exist positive number ε and $\boldsymbol{R}(\lambda) \in \mathscr{A}(D_\varepsilon; \mathscr{Y})$ where $D_\varepsilon = \{\lambda \in C; Re\lambda \geq 0, 0 < |\lambda| \leq \varepsilon\}$ such that $\boldsymbol{R}(\lambda)\boldsymbol{F} = (\lambda + \boldsymbol{A})^{-1}\boldsymbol{F}$,*

$$\|(\frac{d}{d\lambda})^k \boldsymbol{R}(\lambda)\boldsymbol{F}\|_{X^m_q(\Omega_b)} + \|(\frac{d}{d\lambda})^k \boldsymbol{PR}(\lambda)\boldsymbol{F}\|_{2+m,q,\Omega_b} \leq C(q, b, k, \epsilon, m) max\{1, |\lambda|^{1/2-k}\}\|\boldsymbol{F}\|_{X^m_q(\Omega_b)},$$

for any $\lambda \in D_\epsilon$, $\boldsymbol{F} \in \boldsymbol{Y}^m_{q,b}(\Omega)$ and $k, m \geq 0$ integers.

Proof. The results for the case $m = 0$ were proved by Kobayashi [4]. When $m \geq 1$, we can prove by employing the same argument as in Kobayashi [4]. In fact, we shall investigate the parametrix which was constructed in [4]. First we consider the following stationary equations in $\mathbf{R}^3$ with a complex parameter λ

(2.3) $(\lambda + \boldsymbol{A})\boldsymbol{U} = \boldsymbol{F}$ in $\mathbf{R}^3$.

By taking Fourier transform on (2.3) we obtain $[\lambda + \hat{\boldsymbol{A}}(\xi)]\hat{\boldsymbol{U}} = \hat{\boldsymbol{F}}$, where $\mathscr{F}(f) = \hat{f}$ stand for the Fourier transforms of f. Here $\hat{\boldsymbol{A}}$ is the 5×5 symmetric matrix as follows:

$$\hat{\boldsymbol{A}}(\xi) = \begin{pmatrix} 0 & i\gamma\xi_k & 0 \\ i\gamma\xi_j & \delta_{jk}\alpha|\xi|^2 + \beta\xi_j\xi_k & i\omega\xi_j \\ 0 & i\omega\xi_k & \kappa|\xi|^2 \end{pmatrix}$$

where $i = \sqrt{-1}$ and $\delta_{jk} = 0$ when $k \neq j$ and $= 1$ when $k = j$. Set for $\boldsymbol{F} \in \boldsymbol{X}_q(\mathbf{R}^3)$

$$(2.4) \quad \boldsymbol{R}_0(\lambda)\boldsymbol{F}(x) = {}^T(R_{0,\rho}(\lambda)\boldsymbol{F}(x), \mathbf{R}_{0,v}(\lambda)\boldsymbol{F}(x), R_{0,\theta}(\lambda)\boldsymbol{F}(x)) = \mathscr{F}^{-1}\{[\lambda + \hat{\boldsymbol{A}}(\xi)]^{-1}\hat{\boldsymbol{F}}(\xi)\}(x).$$

Then we have the following estimates: Let $1 < q < \infty$, b be a positive number. Then for ${}^\forall\boldsymbol{F} \in \boldsymbol{X}^m_q(\mathbf{R}^3)$ with $\boldsymbol{F}(x) = 0$ for $x \in \mathbf{R}^3 \setminus B_b$ and ${}^\forall\lambda \in D_\epsilon$

$$(2.5) \quad \|(\frac{d}{d\lambda})^k \boldsymbol{R}_0(\lambda)\boldsymbol{F}\|_{X^m_q(B_b)} + \|(\frac{d}{d\lambda})^k \boldsymbol{PR}_0(\lambda)\boldsymbol{F}\|_{2+m,q,B_b} \leq C\max\{1, |\lambda|^{1/2-k}\}\|\boldsymbol{F}\|_{X^m_q(\mathbf{R}^3)},$$

where $k, m \geq 0$ are integers and $C = C(\epsilon, q, b, k, m)$ is a constant. Moreover, for $0 < \delta < 1/2$ and $\lambda \in D_\epsilon$

$$(2.6) \quad \|{}^T\boldsymbol{R}_0(\lambda)\boldsymbol{F} - {}^T\boldsymbol{R}_0(0)\boldsymbol{F}\|_{W^{m+1}_q(B_b) \times W^{m+2}_q(B_b) \times W^{m+2}_q(B_b)} \leq C(\epsilon, \delta, q, m, b)|\lambda|^\delta\|\boldsymbol{F}\|_{X^m_q(\mathbf{R}^3)}.$$

In fact, since $\partial^\alpha_x\partial^\beta_x\{\mathbf{R}_{0,v}(\lambda), R_{0,\theta}(\lambda)\}\boldsymbol{F} = \partial^\alpha_x\{\mathbf{R}_{0,v}(\lambda), R_{0,\theta}(\lambda)\}\partial^\beta_x\boldsymbol{F}$ where $|\alpha| \leq 2$, $|\beta| \leq m$ and since $\partial^\alpha_x\partial^\beta_x \mathrm{R}_{0,\rho}(\lambda)\boldsymbol{F} = \partial^\alpha_x \mathrm{R}_{0,\rho}(\lambda)\partial^\beta_x\boldsymbol{F}$ where $|\alpha| \leq 1$, $|\beta| \leq m$, it follows from the estimates (2.5) and (2.6) with $m = 0$ which were proved by Kobayashi [4] that the estimates (2.5) and (2.6) with $m \geq 1$ hold.

Next, let $\boldsymbol{G} \in \boldsymbol{Y}^m_{q,b}(\Omega)$, and let $\boldsymbol{W} \in W^{m+1}_q(\Omega_b) \times \mathbf{W}^{m+2}_q(\Omega_b) \times W^{m+2}_q(\Omega_b)$ be the solution to the problem

$$\boldsymbol{AW} = \boldsymbol{G} \text{ in } \Omega_b, \quad \boldsymbol{PW} = 0 \text{ on } \partial\Omega_b.$$

The existence of such $\boldsymbol{W}$ is guaranteed by Cattabriga [1]. In terms of $\boldsymbol{W}$, let us define the operator $\boldsymbol{L}(0)$ by the relations:

$$\boldsymbol{W} = \boldsymbol{L}(0)\boldsymbol{G} = \{L_p(0)\boldsymbol{G}, \boldsymbol{L}_v(0)\boldsymbol{G}, L_\theta(0)\boldsymbol{G}\}.$$

Here, note that by Cattagriga [1] we have the following estimates for any $\boldsymbol{G} \in \boldsymbol{Y}^m_{q,b}(\Omega)$

$$(2.7) \quad \|\boldsymbol{L}(0)\boldsymbol{G}\|_{X^m_q(\Omega_b)} + \|\boldsymbol{PL}(0)\boldsymbol{G}\|_{m+2,q,\Omega_b} \leq C(q, b)\|\boldsymbol{G}\|_{X^m_q(\Omega_b)},$$

and $L_\rho(0)\boldsymbol{G}$ is unique up to an additive constant.

Now, let b be a fixed constant $b > R_0 + 3$. Choosing φ in $C^\infty(\mathbf{R}^3)$ so that $\varphi(x) = 1$ for $|x| \geq b - 1$ and $= 0$ if $|x| \leq b - 2$ and choosing $\psi \in C_0^\infty(\Omega_b)$ so that $\int_{\Omega_b} \psi(x)dx = 1$, define the operator $\boldsymbol{R}_1(\lambda)$ and $\boldsymbol{S}(\lambda)$ by the relations: For $\boldsymbol{F} \in \boldsymbol{Y}_{q,b}^m(\Omega)$ and $\lambda \in D_\epsilon \cup \{0\}$

$$(2.8)\quad \boldsymbol{R}_1(\lambda)\boldsymbol{F} = \varphi\boldsymbol{R}_0(\lambda)\boldsymbol{F}_0 + (1-\varphi)\boldsymbol{L}(0)\boldsymbol{F} - \frac{1}{\lambda}\int_{\Omega_b} S(\lambda)\boldsymbol{F}dx\varphi^T(1, 0, 0, 0, 0),$$

$$\boldsymbol{S}(\lambda)\boldsymbol{F} = {}^T\{S_\rho(\lambda)\boldsymbol{F}, \mathbf{S}_v(\lambda)\boldsymbol{F}, S_\theta(\lambda)\boldsymbol{F}\},$$

where $\boldsymbol{F}_0(x) = \boldsymbol{F}(x)$ for $x \in \Omega$ and $= 0$ for $x \in \mathbf{R}^3\backslash\Omega$,

$$S(\lambda)\boldsymbol{F} = \lambda(1-\varphi)L_\rho(0)\boldsymbol{F} + \gamma\nabla\varphi[\mathbf{R}_{0,v}(\lambda)\boldsymbol{F}_0 - \mathbf{L}_v(0)\boldsymbol{F}],$$

$$S_\rho(\lambda)\boldsymbol{F} = S(\lambda)\boldsymbol{F} - \int_{\Omega_b} S(\lambda)\boldsymbol{F}dx\psi,$$

$$\mathbf{S}_v(\lambda)\boldsymbol{F} = \lambda(1-\varphi)\mathbf{L}_v(0)\boldsymbol{F} - \alpha[\Delta\varphi + 2(\partial_j\varphi)\partial_j][\mathbf{R}_{0,v}(\lambda)\boldsymbol{F}_0 - \mathbf{L}_v(0)\boldsymbol{F}] - \beta\nabla\{\partial_j\varphi[\mathbf{R}_{0,v}(\lambda)\boldsymbol{F}_0 - \mathbf{L}_v(0)\boldsymbol{F}]_j\} - \beta\nabla\varphi\{\operatorname{div}[\mathbf{R}_{0,v}(\lambda)\boldsymbol{F}_0 - \mathbf{L}_v(0)\boldsymbol{F}]\} + \gamma\nabla\varphi[R_{0,\rho}(\lambda)\boldsymbol{F}_0 - L_\rho(0)\boldsymbol{F}] + \omega\partial_j\varphi[R_{0,\theta}(\lambda)\boldsymbol{F}_0 - L_\theta(0)\boldsymbol{F}]_j - \frac{\gamma}{\lambda}\int_{\Omega_b} S(\lambda)\boldsymbol{F}dx\psi,$$

$$S_\theta(\lambda)\boldsymbol{F} = \lambda(1-\varphi)L_\theta(0)\boldsymbol{F} - \kappa[\Delta\varphi + 2\partial_j\varphi\partial_j][R_{0,\theta}(\lambda)\boldsymbol{F}_0 - L_\theta(0)\boldsymbol{F}] + \omega\partial_j\varphi[\mathbf{R}_{0,v}(\lambda)\boldsymbol{F}_0 - \mathbf{L}_v(0)\boldsymbol{F}]_j.$$

Since $L_\rho(0)\boldsymbol{F}$ is unique up to additive constant, we may choose $L_\rho(0)\boldsymbol{F}$ in such a way that

$$(2.9)\quad \int_{\Omega_b}(1-\varphi)L_\rho(0)\boldsymbol{F}dx = \int_{B_b} R_{0,\rho}(0)\boldsymbol{F}_0dx - \int_{\Omega_b}\varphi R_{0,\rho}(0)\boldsymbol{F}_0dx.$$

Note that the Stokes formula and (2.9) implies that

$$\int_{\Omega_b} S(\lambda)\boldsymbol{F}dx = \lambda\int_{\Omega_b}(1-\varphi)L_\rho(0)dx\boldsymbol{F} + \int_{B_b}\gamma\operatorname{div}\mathbf{R}_{0,v}(\lambda)\boldsymbol{F}_0dx - \int_{\Omega_b}\varphi\gamma\operatorname{div}[\mathbf{R}_{0,v}(\lambda)\boldsymbol{F}_0 - \mathbf{L}_v(0)\boldsymbol{F}]dx$$

$$= \lambda\{\int_{\Omega_b}(1-\varphi)L_\rho(0)\boldsymbol{F}dx - \int_{B_b} R_{0,\rho}(\lambda)\boldsymbol{F}_0dx + \int_{\Omega_b}\varphi R_{0,\rho}(\lambda)\boldsymbol{F}_0dx\}.$$

It follows from (2.4), (2.5), (2.6), (2.7), (2.8), and (2.9) that

$$\boldsymbol{R}_1(\lambda) \in \mathscr{A}(D_\epsilon; \mathscr{Y}),\quad {}^T\boldsymbol{R}_1(0) \in \mathscr{B}(\boldsymbol{Y}_{q,b}^m(\Omega), W_{q,loc}^{m+1}(\Omega) \times \mathbf{W}_{q,loc}^{m+2}(\Omega) \times W_{q,loc}^{m+2}(\Omega)),$$

$$(2.10)\quad (\lambda + \boldsymbol{A})\boldsymbol{R}_1(\lambda)\boldsymbol{F} = (1 + \boldsymbol{S}(\lambda))\boldsymbol{F} \text{ in } \Omega,\quad \boldsymbol{P}\boldsymbol{R}_1(\lambda)\boldsymbol{F} = 0 \text{ on } \partial\Omega,$$

$$\boldsymbol{S}(0) \in \mathscr{B}(\boldsymbol{Y}_{q,b}^m(\Omega), \boldsymbol{X}_q^{m+1}(\Omega)),\ \boldsymbol{S}(\lambda) \in \mathscr{B}(\boldsymbol{Y}_{q,b}^m(\Omega), \{\boldsymbol{W}_q^{m+1}(\Omega)\}^5) \text{ for any } \lambda \in D_\epsilon.$$

Also we have $\int_{\Omega_b} S_\rho(\lambda)\boldsymbol{F}dx = 0$ for $\lambda \in D_\epsilon \cup \{0\}$ and

$$(2.11)\quad \|\boldsymbol{S}(\lambda) - \boldsymbol{S}(0)\|_{\mathscr{B}(\mathrm{Y}_{q,b}^m(\Omega),\mathrm{Y}_{q,b}^m(\Omega))} \leq C(q, b, \delta)|\lambda|^\delta$$

for $\lambda \in D_\epsilon$ where $0 < \delta < 1/2$. Noting that supp $\boldsymbol{S}(0)\boldsymbol{F}$ is contained in Ω_b, it follows from (2.11) and Rellich's compactness theorem that $\boldsymbol{S}(0)$ is a compact operator from $\boldsymbol{Y}_{q,b}^1(\Omega)$ into itself. Since $1 + \boldsymbol{S}(0)$ is injective in $\mathscr{B}(\boldsymbol{Y}_{q,b}(\Omega), \boldsymbol{Y}_{q,b}(\Omega))$ by Lemma 4.6 in Kobayashi [4], by Fredholm's alternative theorem, $1 + \boldsymbol{S}(0) \in \mathscr{B}(\boldsymbol{Y}_{q,b}^m(\Omega), \boldsymbol{Y}_{q,b}^m(\Omega))$ has the bounded inverse $(1 + \boldsymbol{S}(0))^{-1}$. Thus putting $\|(1 + \boldsymbol{S}(0))^{-1}\|_{\mathscr{B}(\mathrm{Y}_{q,b}^m(\Omega),\mathrm{Y}_{q,b}^m(\Omega))} = M$, by (2.11), there exists an $\epsilon > 0$ such that $1 + \boldsymbol{S}(\lambda)$ also has the bounded inverse $(1 + \boldsymbol{S}(\lambda))^{-1}$ from $\boldsymbol{Y}_{q,b}^m(\Omega)$ onto itself whenever $\lambda \in D_\epsilon$, and moreover

$$(2.12)\quad \|(1 + \boldsymbol{S}(\lambda))^{-1}\|_{\mathscr{B}(\mathrm{Y}_{q,b}^m(\Omega),\mathrm{Y}_{q,b}^m(\Omega))} \leq 2M \text{ for } \lambda \in D_\epsilon.$$

It follows from (2.5), (2.7), (2.8), and (2.10) that for $\boldsymbol{F} \in \boldsymbol{Y}_{q,b}^m(\Omega)$, $\lambda \in D_\epsilon$ and $k \geq 0$ integer

$$(2.13)\quad \|(\frac{d}{d\lambda})^k\boldsymbol{R}_1(\lambda)\boldsymbol{F}\|_{\mathrm{X}_q^m(\Omega_b)} + \|(\frac{d}{d\lambda})^k\boldsymbol{P}\boldsymbol{R}_1(\lambda)\boldsymbol{F}\|_{m+2,q,\Omega_b} \leq C\max\{1, |\lambda|^{1/2-k}\}\|\boldsymbol{F}\|_{\mathrm{X}_q^m(\Omega_b)}.$$

Thus putting $\boldsymbol{R}(\lambda) = \boldsymbol{R}_1(\lambda)(1 + \boldsymbol{S}(\lambda))^{-1}$, combining (2.12) and (2.13) implies Lemma 2.2. ■

Now we shall prove our main theorem. To do this we prepare the following lemma, which was proved by Shibata (see Theorems 3.2 and 3.7 of [10]).

Lemma 2.3. *Let X be a Banach space with norm $|\cdot|_X$. Let $f(\tau)$ be a function of $C^\infty(\boldsymbol{R}\backslash\{0\}: X)$ such that $f(\tau) = 0$, $|\tau| \geq a$ with some $a > 0$. Assume that there exists a constant $C(f)$ depending on f such that for any $0 < |\tau| \leq a$,*

$$|(\frac{d}{d\tau})^k f(\tau)|_X \leq C(f)|\tau|^{-1/2-k},\ k = 0, 1.$$

Put $g(t) = \int_{-\infty}^{\infty} f(\tau)e^{-it\tau}d\tau$. Then

$$|g(t)|_X \leq C(1 + t)^{-1/2}C(f).$$

Let $\boldsymbol{U} \in \boldsymbol{Y}_{q,b}^1(\Omega)$, $b > b_0$ and let $\psi \in C_0^\infty(\boldsymbol{R}^3)$ such that $\psi(x) = 1$ for $|x| \leq b$ and $= 0$ for $|x| \geq b + 1$. Taking $\eta(s) \in C^\infty(\boldsymbol{R})$ so that $\eta(s) = 1$ for $|s| \leq 1/4$ and $= 0$ for $|s| \geq 1/2$ we can represent the semigroup as follows (see Kobayashi [4]):

$$(2.14)\quad \psi e^{-tA}\boldsymbol{U} = \boldsymbol{J}_0(t)\boldsymbol{U} + \boldsymbol{J}_\infty(t)\boldsymbol{U}$$

where

$$\boldsymbol{J}_0(t)\boldsymbol{U} = \frac{1}{2\pi t}(\psi\int_{-\infty}^{\infty} e^{its}\eta(s)\frac{d}{ds}(is + \boldsymbol{A})^{-1}\boldsymbol{U}ds),$$

$$\boldsymbol{J}_\infty(t)\boldsymbol{U} = \frac{1}{2\pi t}(\psi\int_{-\infty}^{\infty} e^{its}(1-\eta(s))\frac{d}{ds}(is + \boldsymbol{A})^{-1}\boldsymbol{U}ds).$$

By (1.2), (2.2), and by Lemma 2.1 we have

$$\|D_x^\alpha(1-\eta(s))(\frac{d}{ds})^N(is+\boldsymbol{A})^{-1}\boldsymbol{U}\|_{q,\Omega}$$

$$(2.15) \quad \le (1-\eta(s))\{\|(is+\boldsymbol{A})^{-N-1}\boldsymbol{U}\|_{X_q^1(\Omega)} + \|\boldsymbol{P}(is+\boldsymbol{A})^{-N-1}\boldsymbol{U}\|_{3,q,\Omega}\}$$

$$\le C(N)(1+|S|)^{-(N-1)/2}\|U\|_{X_q^1(\Omega)},$$

where $D_x^\alpha = (\partial_x^{\alpha 1},\ldots,\partial_x^{\alpha 5})$, $|\alpha_1| \le 2$, $|\alpha_j| \le 3$ ($j = 2,\ldots,5$) and hence by the relation $\frac{1}{t}\cdot\frac{d}{d\lambda}e^{t\lambda} = e^{t\lambda}$, we have

$$(2.16) \quad \|D_x^\alpha\partial_t^M\boldsymbol{J}_\infty(t)\boldsymbol{U}\|_{q,\Omega} \le C(N, M, \alpha)t^{-N}\|\boldsymbol{U}\|_{X_q^1(\Omega)}$$

for any integers $N \ge 2$, $M \ge 0$. On the other hand, noting that

$$D_x^\alpha\partial_t^M\boldsymbol{J}_0(t)\boldsymbol{U} = \frac{1}{2\pi}\sum_{n=0}^{M}\binom{M}{N}\partial_t^{M-N}t^{-1}D_x^\alpha \{\psi\int_{-\infty}^{\infty}e^{its}\eta(s)(is)^n\frac{d}{ds}\boldsymbol{R}(is)\boldsymbol{U}ds\}$$

it follows from Lemma 2.2 and Lemma 2.3 that

$$(2.17) \quad \|D_x^\alpha\partial_t^M\boldsymbol{J}_0(t)\boldsymbol{U}\|_{q,\Omega} \le C(M, b, q)(1+t)^{-(M+3/2)}\|\boldsymbol{U}\|_{X_q^1(\Omega)}$$

for any $\boldsymbol{U} \in \boldsymbol{Y}_{q,b}^1(\Omega)$, integer $M \ge 0$ and $t \ge 1$. Combining (2.15), (2.16), and (2.17) implies Theorem 1.1. This completes the proof.

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