Multifractal Spectrum of Multinomial Measures

By Tatsuya OKADA,*) Takeshi SEKIGUCHI,**) and Yasunobu SHIOTA**)

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1. Introduction. Multifractal theory is one of the most important branch in the fractal theory. There are numerous papers on multifractal theory. See, for example, Cawley and Mauldin [2], Edgar and Mauldin [3], Falconer [4], Feder [5] and Mandelbrot [6]. However, there are few works on the classification of the probability measures by their multifractal spectrum $f(\alpha)$. It seems very hard to investigate such problems for general probability measures. In this paper we deal with multinomial measures, which is the most simple one, and clarify the relationship of two multinomial measures with an identical multifractal spectrum $f(\alpha)$. Throughout this paper we use the convention $0/0 = 0$.

2. Preliminaries. We first summarize some fundamental definitions and results in multifractal theory. Our approach essentially follows that of Falconer [4].

Let $I = I_{0,0} = [0, 1]$ and

$$I_{n,j} = [\frac{j}{p^n}, \frac{j+1}{p^n}), \ j = 0, 1, \ldots, p^n - 2,$$

$$I_{n,p^n-1} = [\frac{p^n - 1}{p^n}, 1]$$

for $n = 1, 2, 3, \ldots$.

Definition 2.1. *Let μ be a probability measure on I. For $-\infty < \alpha < \infty$ and $-\infty < q < \infty$, we define*

$$N_{p^{-n}}(\alpha) = \#\{I_{n,j} : \mu(I_{n,j}) \geq (p^{-n})^{\alpha}\},$$

$$S_{p^{-n}}(q) = \sum_{\mu(I_{n,j})>0} \mu(I_{n,j})^q$$

and

$$f(\alpha) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{\log\{N_{p^{-n}}(\alpha+\varepsilon) - N_{p^{-n}}(\alpha-\varepsilon)\}}{-\log p^{-n}},$$

$$\tau(q) = \lim_{n\to\infty} \frac{\log S_{p^{-n}}(q)}{-\log p^{-n}}.$$

Here $\log 0 = -\infty$ *We say $f(\alpha)$ the multifractal spectrum of μ and $\tau(q)$ the mass exponent of μ.*

Definition 2.2. *Let $p \geq 2$ be a positive integer and $\mathbf{r} = (r_0, r_1, \ldots, r_{p-1})$ be a random vector such that $0 < r_l < 1$ for $l = 0, 1, \ldots, p-1$. The probability measure $\mu_{p,\mathbf{r}}$ on I defined by*

$$\mu_{p,\mathbf{r}}(I_{n+1,pj+l}) = r_l \mu_{p,\mathbf{r}}(I_{n,j}) \tag{1}$$

for $n = 0, 1, 2, \ldots, j = 0, 1, \ldots, p^n - 1, l = 0, 1, \ldots, p-1$, is said to be a multinomial measure.

The following result is well-known in multifractal theory.

Proposition A. *For every multinomial measure $\mu_{p,\mathbf{r}}$, the multifractal spectrum $f(\alpha)$ exists and satisfies the equality*

$$\tau(q) = \sup_{0\leq\alpha<\infty} \{f(\alpha) - q\alpha\}. \tag{2}$$

For $\mu_{p,\mathbf{r}}$, we have

$$S_{p^{-n}}(q) = \sum (r_0^{\alpha_0} r_1^{\alpha_1} \ldots r_{p-1}^{\alpha_{p-1}})^q \frac{n!}{\alpha_0!\alpha_1!\cdots\alpha_{p-1}!}$$
$$= (r_0^q + r_1^q + \cdots + r_{p-1}^q)^n,$$

and hence

$$\tau(q) = \frac{\log S_{p^{-n}}(q)}{-\log p^{-n}} = \frac{\log(r_0^q + r_1^q + \cdots + r_{p-1}^q)}{\log p}. \tag{3}$$

Further we can calculate the multifractal spectrum $f(\alpha(q))$ by use of the so-called Legendre transform:

$$\alpha(q) = -r'(q) = -\frac{\sum r_i^q \log r_i}{(\sum r_i^q)\log p},$$

$$f(\alpha(q)) = \tau(q) + q\alpha(q) = \frac{\log \sum r_i^q}{\log p} - \frac{q\sum r_i^q \log r_i}{\sum r_i^q \log p}.$$

3. Result. We now state our result. Let $\mu_{p,\mathbf{r}}$ and $\mu_{p',\mathbf{r}'}$ be two multinomial measures. We denote their multifractal spectrums by $f_{p,\mathbf{r}}(\alpha)$ and $f_{p',\mathbf{r}'}(\alpha)$, respectively and also denote their mass exponents by $\tau_{p,\mathbf{r}}(q)$ and $\tau_{p',\mathbf{r}'}(q)$, respectively.

Theorem. *Assume that neither $\mu_{p,\mathbf{r}}$ nor $\mu_{p',\mathbf{r}'}$ is the Lebesgue measure. Then $f_{p,\mathbf{r}}(\alpha) = f_{p',\mathbf{r}'}(\alpha)$ if and only if there exist a unique positive integer u and a unique random vector $\mathbf{s} = (s_0, s_1, \ldots, s_{u-1})$ satisfying the following conditions:*

(C-1) p and p' are expressed by

$$p = u^n, \ p' = u^m$$

with some mutually prime numbers n and m.

(C-2) r_i ($i = 0, 1, \ldots, p-1$) and r'_j ($j = 0, 1, \ldots, p'-1$) are represented in the form

$$r_i = s_{i_1}, s_{i_2}, \ldots, s_{i_n}, \ i_1, i_2, \ldots, i_n \in \{0, 1, \ldots, u-1\},$$

*) Department of Mathematics, Fukushima Medical College.

**) Faculty of Liberal Arts, Tohoku Gakuin University.

$r'_j = s_{j_1} s_{j_2} \dots s_{j_m}$, $j_1, j_2, \dots, j_m \in \{0, 1, \dots, u-1\}$.

The sufficiency immediately follows from the definition of multinomial measures. In the following we prove the necessity. We first notice that the condition $f_{p,r}(\alpha) = f_{p',r'}(\alpha)$ implies $\tau_{p,r} = \tau_{p',r'}$. Hence, by (3), we have

$$\sum_{i=0}^{p-1} r_i^t = \left(\sum_{i=0}^{p'-1} r'^t_i\right)^{\frac{\log p}{\log p'}}$$

We now define

$$g(z) = \sum_{i=0}^{p-1} r_i^z = \sum_{i=0}^{p-1} e^{z \log r_i},\ h(z) = \sum_{i=0}^{p'-1} r'^z_i = \sum_{i=0}^{p'-1} e^{z \log r'_i}$$

for $z \in \boldsymbol{C}$. Evidently g and h are entire functions with no pole and

$$g(z) = \{h(z)\}^{\log p/\log p'}.$$

We first state several elementary lemmas in function theory.

Lemma 3.1. *The functions $g(z)$ and $h(z)$ have (common) zeros and the order of them is equal to* 1.

Proof. We represent the function $g(z)$ in the form

(4) $$g(z) = \sum_i e^{z \log r_i} = \sum \frac{\sum (\log r_i)^n}{n!} z^n = \sum c_n z^n, \text{ say.}$$

Then, denoting the order of $g(z)$ by λ, we have by Stirling's formula

$$\lambda = \limsup_{n\to\infty} \frac{n \log n}{\log|1/c_n|} = \limsup_{n\to\infty} \frac{n\log n}{\log n! - \log \sum (\log r_i)^n} = 1.$$

Hence, if we suppose that $g(z)$ has no zeros, then there exists a polynomial $\alpha_1 + \alpha_2 z$ of degree 1 such that

$$g(z) = e^{\alpha_1 + \alpha_2} = e^{\alpha_1} \sum \frac{\alpha_2^n z^n}{n!}.$$

Comparing with (4), we have

$$e^{\alpha_1} = p,\ \alpha_2^n = \sum_i (\log r_i)^n$$

for every n. Hence we obtain $r_1 = r_2 = \cdots = r_{p-1}$. This contradicts the assumption. ■

Lemma 3.2. $\log p/\log p'$ *is a rational number. Therefore $\{g(z)\}^m = \{h(z)\}^n$ for some positive integers m and n with $(m, n) = 1$.*

Proof. Since $g(z)$ and $h(z)$ are entire functions with zeros but with no pole, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)} dz = n',\quad \frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)} dz = \mathrm{m}'$$

for some positive integers m' and n', where γ is a closed contour not containing zeros on it. Put $\alpha = \log p/\log p'$. Since $g(z) = (h(z))^\alpha$, we have

$$g'(z) = \alpha h'(z)(h(z))^{\alpha-1}$$

and

$$\int_\gamma \frac{g'(z)}{g(z)} dz = \int_\gamma \frac{\alpha h'(z)(h(z))^{\alpha-1}}{(h(z))^\alpha} dz = \alpha \int_\gamma \frac{h'(z)}{h(z)} dz.$$

Hence we obtain $\alpha = m'/n'$. Obviously we can choose positive integers m and n with $(m, n) = 1$ so that $m/n = m'/n'$. ■

Lemma 3.3. *Let $g(z) = g(x+iy) = \sum e^{(x+iy)\log r_i}$. Then there exist $M > 0$ and $c_1, c_2 > 0$ such that*

$$0 < c_1 < |g(x+iy)| < c_2 < \infty$$

for $M < x < M+1$.

Proof. Let $r_{\max} = \max_{0\le k\le p-1} r_k$ and $\tilde{r} = \max_{r_k \ne r_{\max}} r_k$. Since

$$g(x+iy) = e^{x \log r_{\max}} \sum_i e^{x(\log r_i - \log r_{\max})} e^{iy \log r_i},$$

taking M large enough so that $1 > 2pe^{M(\log\tilde{r} - \log r_{\max})}$, there exists $c_1 > 0$ such that $|g(x+iy)| > c_1 > 0$ for $x > M$. Evidently $|g(x+iy)|$ is bounded from above for $M < x < M+1$. ■

Lemma 3.4. *Let $\Omega_1 = \{(x, y) \in \boldsymbol{C} : M < x < M+1\}$. Then $\log g(z)$ is almost periodic in Ω_1.*

Proof. By the definition of $g(z)$, it is clear that $g(z)$ is almost periodic in Ω_1. Since

$$\begin{aligned}\log g(z+i\tau) - \log g(z) &= (\log|g(z+i\tau)| - \log|g(z)|) \\ &\quad + i(\arg g(z+i\tau) - \arg g(z)),\end{aligned}$$

by Lemma 3.3, there exists $c_0 > 0$ such that

$$|\log g(z+\tau i) - \log g(z)| < \varepsilon$$

for z satisfying $|g(z)| > c_0$. Hence $\log g(z)$ is almost periodic. ■

Lemma 3.5. *If $a(z)$ is an almost periodic function on Ω_1 satisfying $|a(z)| < C < \infty$ for $z \in \Omega_1$, then $e^{a(z)}$ is also almost periodic on Ω_1.*

Proof. Since

$$\begin{aligned}e^{a(z+i\tau)} - e^{a(z)} &= e^{\Re a(z+i\tau) + i\Im a(z+i\tau)} - e^{\Re a(z) + i\Im a(z)} \\ &= (e^{\Re a(z+i\tau)} - e^{\Re a(z)}) e^{i\Im a(z+i\tau)} \\ &\quad + e^{\Re a(z)}(e^{i\Im a(z+i\tau)} - e^{i\Im a(z)}),\end{aligned}$$

we have

$$\begin{aligned}|e^{a(z+i\tau)} - e^{a(z)}| &\le |e^{\Re a(z+i\tau)} - e^{\Re a(z)}| \\ &\quad + e^{\Re a(z)} |e^{i(\Im a(z+i\tau) - \Im a(z))} - 1| \\ &\le e^C \varepsilon + 2e^C \sin\varepsilon.\end{aligned}$$

This completes the proof. ■

We now state a key lemma.

Lemma 3.6. *The functions $g(z)$ and $h(z)$ are represented in the form*

(5) $$g(z) = \left(\sum_{j=1}^N a_j e^{-\lambda_j z}\right)^n,$$

(6) $$h(z) = \left(\sum_{j=1}^N a_j e^{-\lambda_j z}\right)^m,$$

where $\lambda_1 < \lambda_2 < \cdots < \lambda_N$.

Proof. By Lemmas 3.1, 3.2 and Hadamard's theorem, g and h are represented in the form

$$g(z) = e^{\alpha_1' + \alpha_2' z} \prod \left(1 - \frac{z}{\beta_i}\right)^{k_i'} e^{\frac{k_i'}{\beta_i} z},$$

$$h(z) = e^{\alpha_1'' + \alpha_2'' z} \prod \left(1 - \frac{z}{\beta_i}\right)^{k_i''} e^{\frac{k_i''}{\beta_i} z}.$$

Furthermore, since zeros of g coincide with those of h, we have

$$\alpha_1' m = \alpha_1'' n, \quad \alpha_2' m = \alpha_2'' n, \quad k_i' m = k_i'' n.$$

Hence

$$g(z) = (I(z))^n, \quad h(z) = (I(z))^m \tag{7}$$

where

$$I(z) = e^{\alpha_1 + \alpha_2 z} \prod \left(1 - \frac{z}{\beta_i}\right)^{k_i} e^{\frac{k_i}{\beta_i} z}$$

for some α_1, α_2, and k_i, $i = 1, 2, \ldots$. By Lemma 3.4, $\log I(z)$ is almost periodic on Ω_1. Hence, by Lemma 3.5, $I(z)$ is also almost periodic on Ω_1. Then it is well-known that there exists a Dirichlet series corresponding to $I(z)$:

$$I(z) \sim \sum_{j=1}^{\infty} a_j e^{-\lambda_j z}, \; a_j = \lim_{\tau \to \infty} \frac{1}{\tau} \int_a^{a+\tau} I(x+iy) e^{i\lambda_j y} dy.$$

By (7) and the fact that $(m, n) = 1$, we have $I(z) = g^{\alpha}(z) h^{\beta}(z)$ for some integers α, β. We also have

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_a^{a+\tau} g^{\alpha}(z) h^{\beta}(z) e^{\lambda z} dz = 0$$

except for a finite set of λ by a theorem of Riemann-Lebesgue's type for almost periodic functions (see [1, p. 22, Lemma]). Therefore we obtain $I(z) = \sum_{j=1}^{N} a_j e^{-\lambda_j z}$ for some positive integer N. ■

Proof of the necessity. It suffices to show that there exists a unique random vector $\boldsymbol{s} = (s_0, s_1, \ldots, s_{u-1})$ such that

$$g(t) = \left(\sum_{j=0}^{u-1} s_j^t\right)^n, \; h(t) = \left(\sum_{j=0}^{u-1} s_j^t\right)^m$$

for all t. We have only to prove in the case

$$0 > \log r_0 \geq \log r_1 \geq \cdots \geq \log r_{p-1}.$$

Multiplying $e^{\lambda_i n t}$ to both sides of (5), we get

$$\sum_{j=1}^{p-1} e^{t(n\lambda_i + \log r_j)} = \left(\sum_{j=1}^{N} a_j e^{(\lambda_i - \lambda_j)t}\right)^n.$$

Therefore,

$$a_i^n = \#\{j : n\lambda_i + \log r_j = 0\}.$$

In the same manner we have by (6)

$$a_i^m = \#\{j : m\lambda_i + \log r_j' = 0\}.$$

Consequently, noticing that n and m are relatively prime, we know that a_i is a positive integer. Putting $z = 0$ in (5) and (6), we have $p = (\sum a_j)^n$ and $p' = (\sum a_j)^m$, respectively. Let $u = \sum a_i$. Evidently $p = u^n$, $p' = u^m$ and

$$I(t) = \sum_{j=1}^{N} a_j e^{-\lambda_j t} = \sum_{i=0}^{u-1} e^{-\lambda_i' t},$$

where $\lambda_i' = \lambda_j$ for some j. Let $-\lambda_j' = \log s_j$, then we know that $0 < s_j < 1$, $\sum_{j=0}^{u-1} s_j = 1$ and

$$g(t) = \left(\sum_{j=0}^{u-1} e^{t \log s_j}\right)^n = \left(\sum s_j^t\right)^n,$$

$$h(t) = \left(\sum_{j=0}^{u-1} e^{t \log s_j}\right)^m = \left(\sum s_j^t\right)^m,$$

from which, the necessity follows.

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