Commutant Algebra of Superderivations on a Grassmann Algebra

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Introduction. In his classical book [4], Weyl gave the constructions of the representations of the general linear groups by using Young's symmetrizers. Undoubtedly his theory is very important in the representation theory and many successors have worked in the generalization of this theory. We also try to get a similar construction for natural representations of Cartan-type Lie algabras and Cartan-type Lie superalgabras. As the first step in this direction, it seems necessary to calculate the commutant algebra of this representation. For the case of Cartan-type Lie algebra of vector fields, the first author successfully found the commutant algebra for the case $m \leq n$ (see [2]), where m is the power of tensor product and n is the rank of the Lie algebra. In this article, we want to look for the commutant algebra of the natural representation of Catan-type Lie *superalgebra* $W(n)$ consisting of all the superderivations on the Grassmann algebra of n-variables (see below for the definition). For the case $m \leq n$ (here also m is the power of tensor product), using the same method as in [2], we obtain the result (see Section 2). For the case $m > n$, it seems more complicated, but for $n = 1$ and arbitrary m, we get the similar result as in the case $m \leq n$; furthermore, in this case we also get the bicommutant algebra (see Section 3). For the general case, we conjecture that the result is the same as in the case $m \leq n$. As an evidence, in Section 4, we give an example for the case $n = 2$, $m = 3$.

1. Lie superalgebra $W(n)$ and its natural representation. Let $\Lambda(n)$ be a Grassmann algebra over $\boldsymbol{C}$ in n variables $\xi_1, \xi_2, \cdots, \xi_n$ and Λ_k be the space of k-homogeneous elements of $\Lambda(n)$. Put $\Lambda(n)_{\bar{0}} := \sum_{k:\text{even}} \Lambda_k$ and $\Lambda(n)_{\bar{1}} := \sum_{k:\text{odd}} \Lambda_k$, then $\Lambda(n)$ has a natural $\boldsymbol{Z}_2$-grading and so we consider $\Lambda(n)$ as a superalgebra. Let $W(n)$ be the set of all the superderivations over $\Lambda(n)$, then it becomes naturally a Lie superalgebra. According to the results in [1], every superderivation $D \in W(n)$ can be written in the form $D = \sum_{i=1}^{n} P_i \frac{\partial}{\partial \xi_i}$ with $P_i \in \Lambda(n)$ $(1 \leq i \leq n)$, where $\frac{\partial}{\partial \xi_i}$ is a superderivation of degree 1 defined by $\frac{\partial}{\partial \xi_i} \xi_j = \delta_{ij}$. By definition, the Lie superalgebra $W(n)$ acts on Grassman algebra $\Lambda(n)$ as follows: for any homogeneous $D \in W(n)$ and $\forall\, \xi_{i_1} \wedge \cdots \wedge \xi_{i_r}$,

$$D(\xi_{i_1} \wedge \cdots \wedge \xi_{i_r}) = \sum_{s=1}^{r} (-1)^{(s-1)\deg D} \xi_{i_1} \wedge \cdots \wedge D(\xi_{i_s}) \wedge \cdots \wedge \xi_{i_r}.$$

We call it a *natural representation* of $W(n)$, and denote it by ψ.

Let us consider m-fold tensor product $\otimes^m \Lambda(n)$. Then we have a natural isomorphism as $W(n)$-modules

$$\otimes^m \Lambda(n) \simeq \Lambda[\xi_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq m] =: \Lambda(n, m),$$

where $\Lambda[\xi_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq m]$ is a Grassmann algebra generated by ξ_{ij} $(1 \leq i \leq n,\ 1 \leq j \leq m)$. In the following, we identify $\otimes^m \Lambda(n)$ with $\Lambda(n, m)$. By means of a tensor product, $W(n)$ is imbedded into $\mathrm{End}(\otimes^m \Lambda(n)) \cong \mathrm{End}\Lambda(n, m)$. More precisely, an element

$$D = \sum_{i=1}^{n} P_i(\xi_1, \cdots, \xi_n) \frac{\partial}{\partial \xi_i} \in W(n)$$

corresponds to an element

$$\psi^{\otimes m}(D) = \sum_{i=1}^{n} \sum_{\alpha=1}^{m} P_i(\xi_{1\alpha}, \cdots, \xi_{n\alpha}) \frac{\partial}{\partial \xi_{i\alpha}} \in \mathrm{Der}\Lambda(n, m)$$

via m-fold tensor product $\psi^{\otimes m}$ of ψ.

Let $\mathscr{C}_m$ denote the commutant algebra of $\psi^{\otimes m}(W(n))$ in $\mathrm{End}(\Lambda(n, m))$:

$$\mathscr{C}_m = \{E \in \mathrm{End}(\otimes^m \Lambda(n)) \mid [E, D] = 0,$$

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$$\forall D \in \phi^{\otimes m}(W(n))\}.$$

Then $\mathscr{C}_m$ has a natural $\boldsymbol{Z}_2$-graded structure, $\mathscr{C}_m = \mathscr{C}_{m,\bar{0}} \oplus \mathscr{C}_{m,\bar{1}}$. However, $\mathscr{C}_{m,\bar{1}}$ vanishes as we see in the following lemma.

Lemma 1.1. *The odd subspace $\mathscr{C}_{m,\bar{1}}$ of $\mathscr{C}_m$ vanishes*: $\mathscr{C}_{m,\bar{1}} = \{0\}$.

By this lemma, we have $\mathscr{C}_m = \mathscr{C}_{m,\bar{1}}$ hence the commutant algebra $\mathscr{C}_m$ becomes

$$\mathscr{C}_m = \{E \in \mathrm{End}(\otimes^m \Lambda(n)) \mid ED = DE, \\ \forall D \in \phi^{\otimes m}(W(n))\}.$$

Denote by $[m]$ the set $\{1,2,\cdots, m\}$ of integers, and put $\mathrm{End}[m] = \{\varphi : [m] \to [m]\}$ the set of all the maps from $[m]$ to itself. By composition of maps, $\mathrm{End}[m]$ becomes a semigroup with unit, whose group elements form a permutation group $\mathfrak{S}_m$ of degree m. We call it a *permutation semigroup*. Denote the semigroup ring of $\mathrm{End}[m]$ by $\mathfrak{E}_m$. An element $\varphi \in \mathrm{End}[m]$ acts on $\Lambda(n, m)$ as $(\varphi P)(\xi_{i,j}) = P(\xi_{i,\varphi(j)})\ (P \in \Lambda(n, m))$ and we extend it to $\mathfrak{E}_m$ by linearity (see [2]), thus, we have a representation of $\mathfrak{E}_m$ on $\Lambda(n, m)$. Denote the image algebra of this representation by $\mathscr{E}_m \subset \mathrm{End}\Lambda(n, m)$. The following lemma is easy to prove.

Lemma 1.2. *For arbitrary n and m, we have* $\mathscr{E}_m \subseteq \mathscr{C}_m$.

2. Commutant algebra of $\phi^{\otimes m}(W(n))$ (the case $m \leq n$). Let $(\phi, \Lambda(n))$ be the natural representation of $W(n)$ and $(\phi^{\otimes m}, \Lambda(n, m))$ its m-fold tensor product. Denote by $U(W(n))$ the universal enveloping algebra of $W(n)$, then we have

Lemma 2.1. *Put $\xi(\alpha) = (\xi_{1\alpha}, \cdots, \xi_{n\alpha})$. Then the subalgebra $\phi^{\otimes m}(U(W(n)))$ in $\mathrm{End}\Lambda(n, m)$ is generated by*

$$\sum_{1 \leq \alpha_1, \cdots, \alpha_k \leq m} P_1(\xi(\alpha_1)) \cdots P_k(\xi(\alpha_k)) \frac{\partial^k}{\partial \xi_{b_1\alpha_1} \cdots \partial \xi_{b_k\alpha_k}},$$

where $1 \leq k$ and $1 \leq b_1, \cdots, b_k \leq n$ are indices, P_i is a Grassmannian polynomial in n-variables, and $P_i(\xi(\alpha)) = P_i(\xi_{1\alpha}, \cdots, \xi_{n\alpha})$.

Lemma 2.2. *If $m \leq n$, then the representation of $\mathfrak{E}_m$ on $\Lambda(n, m)$ is faithful, hence we have*

$$\dim \mathscr{E}_m = \dim \mathfrak{E}_m = m^m.$$

Note that the condition $m \leq n$ is necessary. In fact, in Sections 3 and 4, we will give examples where the representation of $\mathfrak{E}_m$ on $\Lambda(n, m)$ is *not* faithful.

Using Lemmas 2.1 and 2.2, we can prove the following

Theorem 2.3. *If $m \leq n$, then the commutant algebra $\mathscr{C}_m$ of $\phi^{\otimes m}(W(n))$ coincides with the representation image $\mathscr{E}_m$ of the semigroup ring $\mathfrak{E}_m$ of the permutation semigroup* $\mathrm{End}[m]$:

$$\mathscr{C}_m = \mathscr{E}_m.$$

Proof. Take an $E \in \mathscr{C}_m$. For Grassmannian polynomials $P_1, \cdots, P_m$ in n-variables, put

$$X(P_1, P_2, \cdots, P_m) = \sum_{1 \leq \alpha_1, \cdots, \alpha_m \leq m} P_1(\xi(\alpha_1)) \cdots P_m(\xi(\alpha_m)) \frac{\partial^m}{\partial \xi_{b_1\alpha_1} \cdots \partial \xi_{bm\alpha_m}},$$

which is in $\phi^{\otimes m}(U(W(n)))$ by Lemma 2.1. Then we have

$$\begin{aligned} &E(P_1(\xi(1))P_2(\xi(2)) \cdots P_m(\xi(m))) \\ &= EX(P_1, P_2, \cdots, P_m)(\xi_{11} \wedge \xi_{22} \wedge \cdots \wedge \xi_{mm}) \\ &= X(P_1, P_2, \cdots, P_m)E(\xi_{11} \wedge \xi_{22} \wedge \cdots \wedge \xi_{mm}). \end{aligned}$$

Since $\Lambda(n, m)$ is generated by $\{P_1(\xi(1)) \cdots P_m(\xi(m)) \mid P_i \in \Lambda(m)\}$, E is completely determined by $E(\xi_{11} \wedge \xi_{22} \wedge \cdots \wedge \xi_{mm})$. On the other hand, Euler operators

$$\sum_{\alpha=1}^{m} \xi_{j\alpha} \frac{\partial}{\partial \xi_{j\alpha}} \quad (1 \leq j \leq n)$$

are contained in $\phi^{\otimes m}(W(n))$, and

$$\begin{aligned} &\sum_{\alpha=1}^{m} \xi_{j\alpha} \frac{\partial}{\partial \xi_{j\alpha}} E(\xi_{11} \wedge \cdots \wedge \xi_{mm}) \\ &= E\Big(\sum_{\alpha=1}^{m} \xi_{j\alpha} \frac{\partial}{\partial \xi_{j\alpha}} (\xi_{11} \wedge \cdots \wedge \xi_{mm})\Big) \end{aligned}$$

$$(2.1) \quad = \begin{cases} E(\xi_{11} \wedge \cdots \wedge \xi_{mm}) & \text{if } 1 \leq j \leq m, \\ 0 & \text{if } m < j \leq n. \end{cases}$$

This means that if $1 \leq j \leq m$, then $E(\xi_{11} \wedge \cdots \wedge \xi_{mm})$ is the eigenvector of the Euler operator with eigenvalue 1 and if $m + 1 \leq j \leq n$, then $E(\xi_{11} \wedge \cdots \wedge \xi_{mm})$ is in the kernel of the Euler operator. So $E(\xi_{11} \wedge \cdots \wedge \xi_{mm})$ is of degree 1 in $(\xi_{j1}, \cdots, \xi_{jm})$ if $1 \leq j \leq m$, and degree 0 in $(\xi_{j1}, \cdots, \xi_{jm})$ if $m + 1 \leq j \leq n$. Hence we obtain

$$E(\xi_{11} \wedge \cdots \wedge \xi_{mm}) = \sum_{1 \leq j_1, \cdots, j_m \leq m} a_{j_1 \cdots j_m} (\xi_{j_1 1} \wedge \cdots \wedge \xi_{j_m m}).$$

So $\dim \mathscr{C}_m$ is less than or equal to m^m.

On the other hand, by Lemma 1.2, $\mathscr{C}_m$ contains the subalgebra $\mathscr{E}_m$ and by Lemma 2.2, its dimension is equal to m^m if $m \leq n$. Therefore we conclude the theorem. Q.E.D.

By the above theorem, we know the structure of the commutant algebra very well for the case $m \leq n$. For the general case, we cannot get a similar result until now. But for the special case $n = 1$ and $n = 2, m = 3$, we obtain the same result as above; furthermore for the case

$n = 1$, we get the bicommutant algebra.

3. Schur duality for $W(1) \times \mathrm{End}[m]$. In this section, we consider the case $n = 1$. In this case, we get a result which is independent of m. For $n = 1$, there holds

$$W(1) = \left\langle \frac{\partial}{\partial \xi}, \xi \frac{\partial}{\partial \xi} \right\rangle, \deg\left(\frac{\partial}{\partial \xi}\right) = 1, \quad \deg\left(\xi \frac{\partial}{\partial \xi}\right) = 0.$$

For convenience, we use the isomorphism $\Lambda(1, m) := \langle \xi_1\ \xi_2, \cdots, \xi_m \rangle \cong \Lambda(m)$. So we have

$$D_{-1} := \phi^{\otimes m}\left(\frac{\partial}{\partial \xi}\right) = \sum_{i=1}^{m} \frac{\partial}{\partial \xi_i},$$
$$D_0 := \phi^{\otimes m}\left(\xi \frac{\partial}{\partial \xi}\right) = \sum_{i=1}^{m} \xi_i \frac{\partial}{\partial \xi_i}.$$

Obviously, $D_{-1}(\Lambda_k) \subseteq \Lambda_{k-1}$, $D_0(\Lambda_k) \subseteq \Lambda_k$ for any k.

Lemma 3.1 *The operator D_{-1} is an exact derivation, i.e., $(D_{-1})^2 = 0$ and the chain complex*

$$0 \to \Lambda_m \xrightarrow{D_{-1}} \Lambda_{m-1} \xrightarrow{D_{-1}} \cdots \xrightarrow{D_{-1}} \Lambda_2 \xrightarrow{D_{-1}} \Lambda_1 \xrightarrow{D_{-1}} \Lambda_0 \to 0$$

is exact.

By the above lemma, we can prove the following theorem.

Theorem 3.2. *Let $n = 1$ and the notations be as above. Then the commutant algebra $\mathscr{C}_m$ of $\phi^{\otimes m}(W(1))$ coincides with the representation image $\mathscr{E}_m$ of semigroup ring $\mathfrak{E}_m$ of the permutation semigroup* $\mathrm{End}[m]$:

$$\mathscr{E}_m = \mathscr{C}_m.$$

Proof. By Lemma 1.2, we have $\mathscr{E}_m \subseteq \mathscr{C}_m$, so it is enough to prove $\mathscr{C}_m \subseteq \mathscr{E}_m$. To do so, we introduce some notations. For any $E \in \mathscr{C}_m$, put $E_k := E|_{\Lambda_k}$, $\mathfrak{I}_k := \{E \in \mathscr{C}_m \mid E_l = 0\ (\forall\, l > k)\}$, and $D_{-1,k} := D_{-1}|_{\Lambda_k}$. Clearly,

$$\mathscr{C}_m = \mathfrak{I}_m \supseteq \mathfrak{I}_{m-1} \supseteq \cdots \supseteq \mathfrak{I}_1 \supseteq \mathfrak{I}_0 = (0),$$

and

$$\mathscr{C}_m \cong (\mathfrak{I}_m/\mathfrak{I}_{m-1}) \oplus (\mathfrak{I}_{m-1}/\mathfrak{I}_{m-2}) \oplus \cdots \oplus \mathfrak{I}_1$$
(as a vector space).

Since Λ_k is decomposed as

$$\Lambda_k = \mathfrak{R}(D_{-1,k+1}) \oplus (\Lambda_{k-1} \wedge \xi_m),$$

we get an isomorphism of vector spaces by using Lemma 3.1:

$$\mathscr{C}_m \cong (\mathfrak{I}_m/\mathfrak{I}_{m-1}) \oplus (\mathfrak{I}_{m-1}/\mathfrak{I}_{m-2}) \oplus \cdots \oplus \mathfrak{I}_1$$
$$\cong \mathrm{Hom}_{\mathbf{C}}(\Lambda_{m-1} \wedge \xi_m, \Lambda_m) \oplus \mathrm{Hom}_{\mathbf{C}}(\Lambda_{m-2} \wedge \xi_m, \Lambda_{m-1}) \oplus \cdots \oplus \mathrm{Hom}_{\mathbf{C}}(\xi_m, \Lambda_1).$$

On the other hand, we can construct a basis of $\mathrm{Hom}_{\mathbf{C}}(\Lambda_{k-1} \wedge \xi_m, \Lambda_k)$, using elements from $\mathscr{E}_m$. So we get a surjection

$$\mathscr{E}_m \to \bigoplus_{k=1}^{m} \mathrm{Hom}_{\mathbf{C}}(\Lambda_{k-1} \wedge \xi_m, \Lambda_k) \simeq \mathscr{C}_m$$

and $\dim \mathscr{C}_m \leq \dim \mathscr{E}_m$. By Lemma 1.2, we have $\mathscr{E}_m = \mathscr{C}_m$. Q.E.D.

From the proof of the above theorem, we can easily know that the dimension of $\mathscr{E}_m$ is $\binom{2m-1}{m-1}$, so the representation of $\mathfrak{E}_m$ on $\Lambda(m)$ is not faithful as indicated in Section 2.

In the special case where $n = 1$, we also get the bicommutant algebra of $\phi^{\otimes m}(W(1))$. The next Theorem 3.3 states that it is the image of the enveloping algebra $\phi^{\otimes m}(U(W(1)))$. Therefore, in this case, we get an analogue of Schur duality for $W(1) \times \mathrm{End}[m]$.

Theorem 3.3. *The bicommutant algebra of m-fold tensor product $\phi^{\otimes m}$ of the natural representation ϕ of $W(1)$ is equal to the image $\phi^{\otimes m}(U(W(1)))$ of the enveloping algebra.*

The proof of this theorem is straight forward, comparing dimensions of $\phi^{\otimes m}(U(W(1)))$ and the bicommutant algebra $\mathscr{C}_m$. See [3] for the detailed proof.

4. Toward the general case. For the general case, we suspect that the commutant algebra $\mathscr{C}_m$ of the representation $\phi^{\otimes m}$ of $W(n)$ is equal to $\mathscr{E}_m$. As an evidence, we give an example for the case $n = 2$, $m = 3$. In this case, since the rank and the dimensions are small, we can calculate out the commutant algebra $\mathscr{C}_3$ explicitly. Let $W(2)$ be a Cartan-type Lie superalgebra of rank 2 as above. We consider 3-fold tensor product $\Lambda(2,3) \cong \otimes^3 \Lambda(2)$ of the natural representation, where $\Lambda(2,3)$ is a Grassmann algebra generated by $\{\xi_{1i}, \xi_{2j} \mid i, j = 1,2,3\}$. By the definition of $\phi^{\otimes 3}$, we have

$$\phi^{\otimes 3}(W(2)) = \langle D_i, D_{ij}, D_{12i} \mid i, j = 1,2 \rangle/_{\mathbf{C}},$$

where

$$D_i = \sum_{\alpha=1}^{3} \frac{\partial}{\partial \xi_{i\alpha}}, \quad D_{ij} = \sum_{\alpha=1}^{3} \xi_{i\alpha} \frac{\partial}{\partial \xi_{j\alpha}},$$
$$D_{12i} = \sum_{\alpha=1}^{3} \xi_{1\alpha} \wedge \xi_{2\alpha} \frac{\partial}{\partial \xi_{i\alpha}}.$$

Put $\Lambda_{p,q} := \Lambda_p(\xi_{1i} \mid 1 \leq i \leq 3) \otimes \Lambda_q(\xi_{2j} \mid 1 \leq j \leq 3)$, where $\Lambda_p(\xi_{1i} \mid 1 \leq i \leq 3)$ (resp. $\Lambda_q(\xi_{2j} \mid 1 \leq j \leq 3)$) is a homogeneous subspace of all the Grassmannian polynomials of degree p (resp. q) generated by $\{\xi_{1i} \mid 1 \leq i \leq 3\}$ (resp. $\{\xi_{2j} \mid 1 \leq j \leq 3\}$), then $\otimes^3 \Lambda(2) = \otimes^3_{p,q=0} \Lambda_{p,q}$. Note that for any $E \in \mathscr{C}_3$, we have $E(\Lambda_{p,q}) \subset \Lambda_{p,q}$ and in the algebra $\phi^{\otimes 3}(W(2))$ we have the following relations:

$$D_i^2 = 0, \ D_{12i}^2 = 0 \ (i = 1,2),$$

$$(D_1 D_{122} + D_{122} D_1) = [D_1, D_{122}] = D_{22},$$
$$(D_2 D_{121} + D_{121} D_2) = [D_2, D_{121}] = D_{11}.$$

Using above relations, we can show that any $E \in \mathscr{C}_3$ is completely determined by $E|_{\Lambda_{2,2}}$ and $E|_{\Lambda_{1,1}}$, and we obtain $\dim \mathscr{C}_3 \leq 24$ after some calculations. On the other hand, also by direct calculation, we have $\dim \mathscr{E}_3 \geq 24$. By the above facts, we have the following theorem.

Theorem 4.1. *Let $n = 2$ and $m = 3$. Then the commutant algebra $\mathscr{C}_3$ of $\phi^{\otimes 3}(W(2))$ coincides with the representation image $\mathscr{E}_3$ of semigroup ring $\mathfrak{E}_3$ of the permutation semigroup* $\mathrm{End}[3] : \mathscr{C}_3 = \mathscr{E}_3$. *The dimension of $\mathscr{C}_3$ is equal to* 24.

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References

[1] V. G. Kac: Lie superalgebra. Advances in Mathematics, **26**, 8–96 (1977).

[2] K. Nishiyama: Commutant algebra and harmonic polynomials of a Lie algebra of vector fields (to appear in J. Alg.).

[3] K. Nishiyama and H. Wang: Commutant algebra of Cartan-type Lie superalgebra $W(n)$ (to appear in J. Math. Kyoto Univ.).

[4] H. Weyl: The Classical Group. Princeton Univ. Press, Princeton, NJ (1946).