

Power Series with the Riemann Zeta-function in the Coefficients

By Masanori KATSURADA^{*)}

Department of Mathematics, Faculty of Science, Kagoshima University

(Communicated by Shokichi IYANAGA, M. J. A., March 12, 1996)

1. Introduction. Let $\zeta(s)$ be the Riemann zeta-function, and $\zeta(s, \alpha)$ with a parameter $\alpha > 0$ the Hurwitz zeta-function defined by

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s} \quad (\text{Re } s > 1),$$

and its meromorphic continuation over the whole s -plane. Let $\Gamma(s)$ be the gamma-function, and $(s)_n = \Gamma(s + n)/\Gamma(s)$ for any integer n Pochhammer's symbol.

The main aim of this note is to investigate two types of power series whose coefficients involve the Riemann zeta-function (see Sections 2 and 3) based on Mellin-Barnes' type integral formulae. Further, as for generalizations of these power series, we shall introduce hypergeometric type generating functions of $\zeta(s)$ and derive their basic properties in the final section. Proofs of the results in the following sections are only sketched. Detailed version of the proofs will appear in a forthcoming paper.

2. Binomial type series. A simple relation

$$\sum_{n=2}^{\infty} \{\zeta(n) - 1\} = 1,$$

which was firstly mentioned by Goldbach in 1729 (see [10, Section 1]), follows immediately from the inversion of the order of the double sum $\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} m^{-n}$. This is in fact derived as a special case of Ramanujan's formula

$$(2.1) \quad \zeta(\nu, 1 + x) = \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} \zeta(\nu + n) (-x)^n$$

for $|x| < 1$ and any complex $\nu \notin \{-1, 0, 1, 2, \dots\}$, which gives a base of his various evaluations of sums involving $\zeta(s)$ (see [7, Sections 5 and 6]). Noting the relations $\zeta(s, 1) = \zeta(s)$ and $(\partial/\partial\alpha)^n \zeta(s, \alpha) = (-1)^n (s)_n \zeta(s + n, \alpha)$, we see that (2.1) is actually the Taylor series expansion of $\zeta(\nu, 1 + x)$ as a function of x near $x = 0$. Srivastava [9][10] proved various summation formulae related to (2.1), while Klusch [6] con-

sidered a generalization of (2.1) to the Lerch zeta-function. This direction has recently been pursued by Yoshimoto, Kanemitsu, and the author [15]. Rane [8] applied (2.1) to study the mean square of Dirichlet L -functions.

For our later purpose we shall prove (2.1) as an application of Mellin-Barnes' type integrals. Suppose first that $\text{Re } \nu > 1$, and set

$$(2.2) \quad F_{\nu}(x) = \frac{1}{2\pi i} \int_{(b)} \frac{\Gamma(\nu + s)\Gamma(-s)}{\Gamma(\nu)} \zeta(\nu + s)x^s ds$$

for $x > 0$, where b is fixed with $1 - \text{Re } \nu < b < 0$, and (b) denotes the vertical straight line from $b - i\infty$ to $b + i\infty$. We can shift the path of integration in (2.2) to the right, provided $0 < x < 1$. Collecting the residues at the poles $s = 0, 1, 2, \dots$ of the integrand, we see that $F_{\nu}(x)$ is equal to the right-hand infinite series in (2.1). On the other hand, since $\zeta(\nu + s) = \sum_{n=1}^{\infty} n^{-\nu-s}$ converges absolutely on the path $\text{Re } s = b$, the term-by-term integration is permissible, and this gives

$$F_{\nu}(x) = \sum_{n=1}^{\infty} (n + x)^{-\nu} = \sum_{n=0}^{\infty} (n + 1 + x)^{-\nu},$$

where each term in the resulting expression could be evaluated by taking $-z = x/n$ in

$$\Gamma(a)(1 - z)^{-a} = \frac{1}{2\pi i} \int_{(\sigma)} \Gamma(a + s)\Gamma(-s)(-z)^s ds$$

for $|\arg(-z)| < \pi$ and $-\text{Re } a < \sigma < 0$ (cf. [14], p. 289, 14.51, Corollary). We therefore obtain (2.1) by analytic continuation.

3. Exponential type series. Chowla and Hawkins [2] found that the sum

$$G_0(x) = \sum_{n=2}^{\infty} \zeta(n) \frac{(-x)^n}{n!}$$

has the asymptotic formula

$$(3.1) \quad G_0(x) = x \log x + (2\gamma - 1)x + \frac{1}{2} + O(e^{-A\sqrt{x}})$$

as $x \rightarrow +\infty$, where γ is Euler's constant and A is a certain positive constant. They conjectured that the error term in (3.1) cannot be essentially sharpened. Let a be an arbitrary fixed real number. Buschman and Srivastava [1] introduced a

^{*)} Research partially supported by Grant-in-Aid for Scientific Research (No. 07740035), Ministry of Education, Science, Sports and Culture, Japan.

more general formulation

$$G_a(x) = \sum_{n>a+1} \zeta(n-a) \frac{(-x)^n}{n!},$$

where n runs through all nonnegative integers with $n > a + 1$, and studied its asymptotic behaviour as $x \rightarrow +\infty$. The special cases $a = -2, -1$ and 1 have been investigated by Tenenbaum [11], Verma [12], and Verma and Prasad [13], respectively.

Let ν be an arbitrary fixed complex number. It is in fact possible to treat a slightly general sum

$$G_\nu(x) = \sum_{n>\operatorname{Re}\nu+1} \zeta(n-\nu) \frac{(-x)^n}{n!},$$

based on the formula

$$(3.2) \quad G_\nu(x) = \frac{1}{2\pi i} \int_{(c)} \Gamma(-s) \zeta(s-\nu) x^s ds$$

for $x > 0$, where c is fixed with $\operatorname{Re} \nu + 1 < c < [\operatorname{Re} \nu] + 2$. Here $[\operatorname{Re} \nu]$ denotes the greatest integer not exceeding $\operatorname{Re} \nu$. (3.2) can be proved by shifting the path (c) to the right, and collecting the residues at the poles $s = [\operatorname{Re} \nu] + 2, [\operatorname{Re} \nu] + 3, \dots$ of the integrand. While the main method of [1] is Euler-Maclaurin's summation device, our treatment of $G_\nu(x)$ is due to a refinement of original [2].

Shifting the path of integration in (3.2) to the left in an appropriate manner, we can show

Theorem 1. *The following formulae hold for all $x \geq 1$.*

(i) If $\nu \notin \{-1, 0, 1, 2, \dots\}$,

$$G_\nu(x) = \Gamma(-\nu-1)x^{\nu+1} - \sum_{n=0}^{[\operatorname{Re}\nu]+1} \zeta(n-\nu) \frac{(-x)^n}{n!} + \mathcal{G}_\nu(x);$$

(ii) If $\nu \in \{-1, 0, 1, 2, \dots\}$,

$$G_\nu(x) = -\frac{(-x)^{\nu+1}}{(\nu+1)!} \left(\log x + 2\gamma - \sum_{n=1}^{\nu+1} \frac{1}{n} \right) - \sum_{n=0}^{\nu} \zeta(n-\nu) \frac{(-x)^n}{n!} + \mathcal{G}_\nu(x),$$

where empty sums are considered to be zero, and $\mathcal{G}_\nu(x)$ is the error term satisfying the estimate

$$(3.3) \quad \mathcal{G}_\nu(x) = O(x^{-C})$$

for any $C > 0$. Here the implied constant depends only on C and ν .

Remark. This theorem gives a refinement of the results in [1].

Chowla and Hawkins suggested in [2] that the error term in (3.1) is expressible in terms of 'almost' Bessel functions, however, it seems that the functions have not been precisely determined.

Let $K_\nu(z)$ be the modified Bessel function of the third kind defined by

$$K_\nu(z) = \frac{\pi}{2 \sin \pi \nu} \{I_{-\nu}(z) - I_\nu(z)\},$$

where $I_\nu(z)$ is the Bessel function with purely imaginary argument (see [4, p. 5, 7.2.2, (12) and (13)]). We can indeed show that $\mathcal{G}_\nu(x)$ has the Voronoï type summation formula involving $K_{\nu+1}(z)$.

Theorem 2. *For any $x \geq 1$, we have*

$$\begin{aligned} \mathcal{G}_\nu(x) &= 2 \left(\frac{x}{2\pi} \right)^{\frac{1}{2}(\nu+1)} \\ &\times \sum_{n=1}^{\infty} n^{-\frac{1}{2}(\nu+1)} \{ e^{-\frac{\pi i}{4}(\nu+1)} K_{\nu+1}(2e^{\frac{\pi i}{4}} \sqrt{2n\pi x}) \\ &+ e^{\frac{\pi i}{4}(\nu+1)} K_{\nu+1}(2e^{-\frac{\pi i}{4}} \sqrt{2n\pi x}) \}. \end{aligned}$$

Let $(\nu, m) = \Gamma\left(\frac{1}{2} + \nu + m\right) / m! \Gamma\left(\frac{1}{2} + \nu - m\right)$

for any integer $m \geq 0$ be Hankel's symbol. Applying an asymptotic expansion for $K_{\nu+1}(z)$ (cf. [4, p. 24, 7.4.1, (4)]) to Theorem 2, we can further prove

Corollary. *The asymptotic formula*

$$\begin{aligned} \mathcal{G}_\nu(x) &= \sqrt{2} \left(\frac{x}{2\pi} \right)^{\frac{1}{2}\nu + \frac{1}{4}} e^{-2\sqrt{\pi x}} \\ &\times \left\{ \sum_{m=0}^{M-1} (\nu+1, m) \cos\left(2\sqrt{\pi x} + \frac{\pi}{4}\right) \right. \\ &\times \left. \left(\nu + \frac{3}{2} + m \right) (32\pi x)^{-\frac{m}{2}} + O(x^{-\frac{M}{2}}) \right\} \end{aligned}$$

holds for all $x \geq 1$ and all integers $M \geq 0$.

Remark. This gives an affirmative answer to the conjecture of Chowla and Hawkins mentioned above.

4. Generating functions of $\zeta(s)$. Let α and ν be arbitrary complex numbers with $\nu \notin \{1, 0, -1, \dots\}$. We define

$$f_\nu(\alpha; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \zeta(\nu+n) z^n \quad (|z| < 1),$$

$$e_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \zeta(\nu+n) z^n \quad (|z| < +\infty).$$

Since $\zeta(\nu+n) \rightarrow 1$ uniformly for $n = 0, 1, 2, \dots$, as $\operatorname{Re} \nu \rightarrow +\infty$, we see that $f_\nu(\alpha; z) \rightarrow (1-z)^{-\alpha}$ and $e_\nu(z) \rightarrow e^z$, as $\operatorname{Re} \nu \rightarrow +\infty$. This suggests us to define the hypergeometric type generating functions of $\zeta(s)$ as

$$(4.1) \quad \mathcal{F}_\nu(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} \zeta(\nu+n) z^n \quad (|z| < 1),$$

$$(4.2) \quad \mathcal{F}_\nu(\alpha; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n n!} \zeta(\nu + n) z^n$$

($|z| < +\infty$),

where α, β and γ are arbitrary fixed complex numbers with $\gamma \notin \{0, -1, -2, \dots\}$. Then we can observe, when $\text{Re } \nu \rightarrow +\infty$, that

$$\begin{aligned} \mathcal{F}_\nu(\alpha, \beta; \gamma; z) &\rightarrow F(\alpha, \beta; \gamma; z), \\ \mathcal{F}_\nu(\alpha; \gamma; z) &\rightarrow F(\alpha; \gamma; z), \end{aligned}$$

where $F(\alpha, \beta; \gamma; z)$ and $F(\alpha; \gamma; z)$ denote hypergeometric functions of Gauss and Kummer, respectively.

Corresponding to Euler's integral formulae for $F(\alpha, \beta; \gamma; z)$ and $F(\alpha; \gamma; z)$ (cf. [3, p. 59, 2.1.3, (10), and p. 255, 6.5, (1)]), we can deduce from term-by-term integration that

$$(4.3) \quad \frac{\Gamma(\beta)\Gamma(\gamma - \beta)}{\Gamma(\gamma)} \mathcal{F}_\nu(\alpha, \beta; \gamma; z) = \int_0^1 \tau^{\beta-1} (1 - \tau)^{\gamma-\beta-1} f_\nu(\alpha; \tau z) d\tau$$

for $0 < \text{Re } \beta < \text{Re } \gamma$ and $|z| < 1$, and

$$(4.4) \quad \frac{\Gamma(\alpha)\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} \mathcal{F}_\nu(\alpha; \gamma; z) = \int_0^1 \tau^{\alpha-1} (1 - \tau)^{\gamma-\alpha-1} e_\nu(\tau z) d\tau$$

for $0 < \text{Re } \alpha < \text{Re } \gamma$ and $|z| < +\infty$. Moreover, corresponding to Mellin-Barnes' integral formulae for $F(\alpha, \beta; \gamma; z)$ and $F(\alpha; \gamma; z)$ (cf. [3, p. 62, 2.1.3, (15), and p. 256, 6.5, (4)]), we can show by the same path shifting argument as in Section 2 that

$$(4.5) \quad \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} \mathcal{F}_\nu(\alpha, \beta; \gamma; z) = \frac{1}{2\pi i} \times \int_{(b)} \frac{\Gamma(\alpha + s)\Gamma(\beta + s)\Gamma(-s)}{\Gamma(\gamma + s)} \zeta(\nu + s) (-z)^s ds$$

for $\text{Re } \nu > 1$, $\max(-\text{Re } \alpha, -\text{Re } \beta, 1 - \text{Re } \nu) < b < 0$ and $|\arg(-z)| < \pi$, and

$$(4.6) \quad \frac{\Gamma(\alpha)}{\Gamma(\gamma)} \mathcal{F}_\nu(\alpha; \gamma; z) = \frac{1}{2\pi i} \times \int_{(c)} \frac{\Gamma(\alpha + s)\Gamma(-s)}{\Gamma(\gamma + s)} \zeta(\nu + s) (-z)^s ds$$

for $\text{Re } \nu > 1$, $\max(-\text{Re } \alpha, 1 - \text{Re } \nu) < c < 0$ and $|\arg(-z)| < \pi/2$.

Formulae (4.1)-(4.6) are fundamental in deriving various properties of $\mathcal{F}_\nu(\alpha, \beta; \gamma; z)$ and $\mathcal{F}_\nu(\alpha; \gamma; z)$. Further investigations and detailed proofs will be given in forthcoming papers.

Acknowledgements. This work was initiated while the author was staying at the Depart-

ment of Mathematics, Keio University (Yokohama). He would like to express his sincere gratitude to this institution, especially to Professor Iekata Shiokawa for his warm hospitality and constant support. He would also like to thank to Professors Aleksandar Ivić, Kohji Matsumoto, and Eiji Yoshida for valuable comments on this work.

References

[1] Buschman, R. G. and Srivastava, H. M.: Asymptotic behavior of some power series with ζ -functions in the coefficients. *Mh. Math.*, **115**, 291–298 (1993).

[2] Chowla, S. and Hawkins, D.: Asymptotic expansions of some series involving the Riemann zeta function. *J. Indian Math. Soc.*, (N. S.), **26**, 115–124 (1962).

[3] Erdélyi, A. *et al.*: Higher Transcendental Functions. vol. I, McGraw-Hill, New York (1953).

[4] Erdélyi, A. *et al.*: ditto. vol. II.

[5] Ivić, A.: The Riemann Zeta-function. John Wiley & Sons, New York (1985).

[6] Klusch, D.: On the Taylor expansion of the Lerch zeta-function. *J. Math. Anal. Appl.*, **170**, 513–523 (1992).

[7] Ramanujan, S.: A series for Euler's constant γ . *Messenger Math.*, **46**, 73–80 (1916–17).

[8] Rane, V. V.: Dirichlet L -function and power series for Hurwitz zeta function. *Proc. Indian Acad. Sci. (Math. Sci.)*, **103**, 27–39 (1993).

[9] Srivastava, H. M.: Summation of a class of series involving the Riemann zeta function. *Rev. Técn. Fac. Ingr. Univ. Zulia (Edición Especial)*, **9**, 79–82 (1986).

[10] Srivastava, H. M.: Sums of certain series of the Riemann zeta function. *J. Math. Anal. Appl.*, **134**, 129–140 (1988).

[11] Tennenbaum, J.: On the function $\sum_{n=0}^{\infty} \frac{\zeta(n+2)}{n!} x^n$. *Math. Scand.*, **41**, 242–248 (1977).

[12] Verma, D. P.: Asymptotic expansion of some series involving the Riemann zeta-function. *Indian J. Math.*, **6**, 121–127 (1964).

[13] Verma, D. P. and Prasad, S. N.: A power series with zeta function coefficients. *J. Bihar Math. Soc.*, **7**, 37–40 (1983).

[14] Whittaker, E. T. and Watson, G. N.: A Course of Modern Analysis. 4th ed., Cambridge University Press, Cambridge (1927).

[15] Yoshimoto, M., Katsurada, M., and Kanemitsu, S.: On the Hurwitz-Lerch zeta-function (preprint).