A Form of Classical Picard Principle

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Let H_d be the fundamental harmonic function on the Euclidean space $\boldsymbol{R}^d$ of dimension $d \geqq 2$, i.e. $H_2(x) = \log(1/|x|)$ and $H_d(x) = 1/|x|^{d-2}$ $(d \geqq 3)$, where $|x| = (\sum_{i=1}^d |x^i|^2)^{1/2}$ is the length of a vector $x = (x^1, \cdots, x^d) \in \boldsymbol{R}^d$. We denote by B^d the unit ball $|x| < 1$ in $\boldsymbol{R}^d$ and by B_0^d the punctured unit ball $0 < |x| < 1$ in $\boldsymbol{R}^d$. Then we have the following

Theorem A. *If $u \geqq 0$ is harmonic in B_0^d $(d \geqq 2)$, then $u = cH_d + v$, where $c \geqq 0$ is a constant and v is harmonic on B^d.*

This result has been called the *Picard principle* by many authors since Bouligand [4] and then Brelot [5] first used the term because of the papers of Picard [10,11] (see also Stòzek [13]); it is also referred to as the *Bôcher theorem* by Helms [6], Wermer [14], and Axler *et al.* [2], etc. since the result is proved by Bôcher [3] 20 years earlier than Picard. Anyway this is one of the results in the potential theory much talked about from various view points: thousands of different proofs have been given to the result; the result is also discussed in the frame of wider degenerate harmonicity such as one given by the Schrödinger equations with potentials having singularities at the origin (cf. e.g. Pinsky [12]); the Martin theory is of course another extension. Recently the following result of Anandam and Damlakhi [1] called our attention:

Theorem B. *Suppose u is harmonic on B_0^2 such that $u(x) \geqq o(|x|^{-s})$ as $|x| \to 0$ with $s \leqq 1$. Then $u = cH_2 + v$, where c is a constant and v is harmonic on B^2.*

This is of course superficially a generalization of Theorem A for $d = 2$. Anandam and Damlakhi [1] proved this in complex analytic way and thus the restriction to the dimension $d = 2$ seems to be essential in their proof. The purpose of this note is to remark that the Fourier expansion method, one of the most standard ways of proving Theorem A, also instantly proves not only Theorem B but also its generalization to higher dimensions. Namely, we will prove the following

1. Theorem. *Suppose u is harmonic in B_0^d such that $u(x) \geqq o(|x|^{-s})$ as $|x| \to 0$ with $s \leqq d - 1$. Then $u = cH_d + v$, where c is a constant and v is harmonic on B^d.*

Proof. We use the polar coordinates $x = r\xi$ for points $x \in \boldsymbol{R}^d \setminus \{0\}$, where $r = |x| > 0$ and $\xi = x/|x| \in S^{d-1} = \partial B^d$. We choose and then fix an orthonormal basis $\{S_{nj} : j = 1, \cdots, N(n)\}$ of the subspace of all spherical harmonics of degree n of $L_2(S^{d-1}, d\sigma)$, where $d\sigma$ is the area element on S^{d-1}. Then $\{S_{nj} : j = 1, \cdots, N(n)\ ; n = 0,1, \cdots\}$ is a complete orthonormal system of $L_2(S^{d-1}, d\sigma)$. We have, as the special case of the addition theorem, $\sum_{j=1}^{N(n)} S_{nj}(\xi)^2 = N(n)/\sigma_d$ on S^{d-1} $(n = 0,1, \cdots)$, where σ_d is the surface area $\sigma(S^{d-1})$ of S^{d-1}. Here $N(0) = 1$ and $N(n) = (2n + d - 2)\Gamma(n + d - 2)/\Gamma(n + 1)\Gamma(d - 1)$ for $n = 1,2, \cdots$. Then we have the following Fourier expansion of $u(r\xi)$ in terms of spherical harmonics $\{S_{nj}\}$:

$$(2)\quad u(r\xi) = \sum_{n=0}^{\infty}\left(\sum_{j=1}^{N(n)} a_{nj}S_{nj}(\xi)\right) r^n + b_{01}H_d(r) + \sum_{n=1}^{\infty}\left(\sum_{j=1}^{N(n)} b_{nj}S_{nj}(\xi)\right) r^{-n-d+2},$$

where $H_d(r) = H_d(x)$ with $|x| = r$ and a_{nj} and b_{nj} $(j = 1, \cdots, N(n)\ ; n = 0,1, \cdots)$ are constants. Here the series on the right hand side of (2) converges uniformly in $\xi \in S^{d-1}$ for any fixed $0 < r < 1$. Multiply $\sqrt{N(n)/\sigma_d} \pm S_{nj}(\xi) \geqq 0$ $(n \geqq 1)$ to both sides of $u(r\xi) \geqq o(r^{-s})$ and

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then integrate both sides of the resulting inequality over S^{d-1} with respect to $d\sigma(\xi)$. (The second and the third author have been using this device frequently in a similar but more general situation (cf. Nakai and Tada [7,8,9]).) Then we obtain

$$\sqrt{N(n)/\sigma_d}\,a_{01} \pm a_{nj}r^n + \sqrt{N(n)/\sigma_d}\,b_{01}\,H_d(r) \pm b_{nj}r^{-n-d+2} \geqq o(r^{-s})$$

or

$$\sqrt{N(n)/\sigma_d}\,a_{01}r^{n+d-2} \pm a_{nj}r^{2n+d-2} + \sqrt{N(n)/\sigma_d}\,b_{01}\,H_d(r)\,r^{n+d-2} \pm b_{nj} \geqq o(r^{n+d-2-s}).$$

Since $n+d-2-s \geqq n+d-2-(d-1) = n-1 \geqq 0$ for $n \geqq 1$, we conclude that $b_{nj} = 0 (j = 1, \cdots, N(n)\,;\, n = 1, 2, \cdots)$. Observe that if we denote the first term on the right hand side of (2) by $v(r\xi)$, then v is harmonic on B^d. Hence on setting $c = b_{01}$ we deduce the desired decomposition $u = cH_d + v$ on B_0^d. □

References

[1] V. Anandam and M. Damlakhi: Bôcher's theorem in $\boldsymbol{R}^2$ and Carathéodory's inequality. Real Analysis Exchange, **19**, 537–539 (1993/4).

[2] S. Axler, P. Bourdon and W. Ramey: Harmonic Function Theory. Springer-Verlag (1992).

[3] M. Bôcher: Singular points of functions which satisfy partial differential equations of the elliptic type. Bull. Amer. Math. Soc., **9**, 455–465 (1903).

[4] M. G. Bouligand: Fonctions Harmoniques. Principes des Picard et de Dirichlet. Mémorial des Science Mathématiques, **11** (1926).

[5] M. Brelot: Étude de l'équation de la chaleur $\Delta u = c(M)u(M)$, $c(M) \geqq 0$, au voisinage d'un point singulier du coefficient. Ann. Éc. Norm., (3), **48**, 153–246 (1931).

[6] L. L. Helms: Introduction to Potential Theory. Wiley-Interscience (1969).

[7] M. Nakai and T. Tada: Picard dimensions of rotation free signed densities. Bull. Nagoya Inst. Tech., **43**, 137–151 (1991) (in Japanese).

[8] M. Nakai and T. Tada: Monotoneity and homogeneity of Picard dimensions for signed radial densities. NIT Sem. Rep. Math., **99**, 3–53 (1993).

[9] M. Nakai and T. Tada: Picard dimensions for potentials of positive type. Bull. Nagoya Inst. Tech., **46**, 139–173 (1994) (in Japanese).

[10] M. É. Picard: Deux théorèmes élémentaires sur les singularités des fonctions harmoniques. C. R. Acad. Sc. Paris, **176**, 933–935 (1923).

[11] M. É. Picard: Sur les singularités des fonctions harmoniques. C. R. Acad. Sc. Paris, **176**, 1025–1026 (1923).

[12] R. G. Pinsky: Positive Harmonic Functions and Diffusion. Cambridge Univ. Press (1995).

[13] W. Stożek: Sur l'allure d'une fonction harmonique dans le voisinage d'un point exceptionnel. Ann. Soc. Polon. Math., **4**, 52–58 (1925).

[14] J. Wermer: Potential Theory. Lect. Notes in Math., vol. 408, Springer-Verlag (1974).