Spectra of the Laplacian with Small Robin Conditional Boundary

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(Communicated by Kiyosi ITÔ, M. J. A., March 12, 1996)

Let M be a bounded domain in $\boldsymbol{R}^4$ with smooth boundary ∂M. Let $\tilde{w}$ be a fixed point in M. Let B_ε be the ball of radius ε with the center $\tilde{w}$. We put $M_\varepsilon = M \setminus \bar{B}_\varepsilon$. Consider the following eigenvalue problem:

$$-\Delta u(x) = \lambda u(x) \qquad x \in M_\varepsilon$$
$$u(x) = 0 \qquad x \in \partial M$$
$$u(x) + k\varepsilon^\sigma \frac{\partial}{\partial \nu_x} u(x) = 0 \quad x \in \partial B_\varepsilon.$$

Here $\sigma \in [0, 1)$ and k denotes a positive constant. Here $\frac{\partial}{\partial \nu_x}$ denotes the derivative along the exterior normal direction with respect to M_ε. Let $\mu_j(\varepsilon)$ be the j th eigenvalue of the above problem.

Let μ_j be the j th eigenvalue of the problem:

$$-\Delta u(x) = \lambda u(x) \quad x \in M$$
$$u(x) = 0 \qquad x \in \partial M.$$

Let $\varphi_j(x)$ be the L^2 normalized eigenfunction of $-\Delta$ associated with μ_j.

We have the following

Theorem 1. *Fix $\sigma \in [0, 1)$. Fix j. Assume that μ_j is a simple eigenvalue. Then,*

$$\mu_j(\varepsilon) - \mu_j = 2\pi^2 k^{-1} \varepsilon^{3-\sigma} \varphi_j(\tilde{w})^2 + O(\varepsilon^{4-2\sigma} + \varepsilon^{3-\sigma}(\varepsilon^{(1/2)+\theta} + \varepsilon^{\sigma+\theta}))$$

for some $\theta > 0$ as ε tends to zero.

The related topics are discussed in Ozawa [1], Roppongi [8], Ozawa-Roppongi [7]. See, for other related topics, Ozawa [2], [3], [4].

The main idea of our Theorem 1 lies in the use of an approximated iterated Green function, which were found in Ozawa [5], [6].

Let $(w_1, \dots, w_4)$ be an orthonormal basis of $\boldsymbol{R}^4$. Then, we put

$$\langle \nabla_w f(x, \tilde{w}), \nabla_w g(\tilde{w}, y)\rangle = \sum_{j=1}^4 \frac{\partial}{\partial w_j} f(x, w) \frac{\partial}{\partial w_j} g(w, y) \Big|_{w=\tilde{w}}$$

Let $G(x, y)$ be the Green function of $-\Delta$ in M under the Dirichlet condition on ∂M. Let $G_\varepsilon(x, y)$ be the Green function of $-\Delta$ in M_ε under the Dirichlet condition on ∂M satisfying

$$G_\varepsilon(x, y) + k\varepsilon^\sigma \frac{\partial}{\partial \nu_x} G_\varepsilon(x, y) = 0 \quad x \in \partial B_\varepsilon.$$

Let $G_\varepsilon^{(2)}(x, y)$ be the iterated Green function of $-\Delta$ which is defined by

$$G_\varepsilon^{(2)}(x, y) = \int_{M_\varepsilon} G_\varepsilon(x, z) G_\varepsilon(z, y) dz.$$

Let $G^{(2)}(x, y)$ be the iterated Green function which is defined by

$$G^{(2)}(x, y) = \int_M G(x, z) G(z, y) dz.$$

We put

$$\begin{aligned} q_\varepsilon(x, y) = G^{(2)}(x, y) &+ g(\varepsilon)(G^{(2)}(x, \tilde{w}) G(\tilde{w}, y) + G(x, \tilde{w}) G^{(2)}(\tilde{w}, y)) \\ &+ h(\varepsilon)(\langle \nabla_w G^{(2)}(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle + \langle \nabla_w G(x, \tilde{w}), \nabla_w G^{(2)}(\tilde{w}, y)\rangle), \end{aligned}$$

where

$$g(\varepsilon) = -((4\pi^2)^{-1}\varepsilon^{-2} + 2^{-1}k\pi^{-2}\varepsilon^{\sigma-3})^{-1}$$
$$h(\varepsilon) = k\varepsilon^\sigma(2^{-1}\pi^{-2}\varepsilon^{-3} + (3/2)k\pi^{-2}\varepsilon^{\sigma-4})^{-1}.$$

Let $\mathbf{G}_\varepsilon^2$, $\mathbf{Q}_\varepsilon$ be the operators defined by

$$(\mathbf{G}_\varepsilon^2 h)(x) = \int_{M_\varepsilon} G_\varepsilon^{(2)}(x, y) h(y) dy$$

and

$$(\mathbf{Q}_\varepsilon h)(x) = \int_{M_\varepsilon} q_\varepsilon(x, y) h(y) dy.$$

We have the following

Proposition 1. *There exists a constant C independent of ε such that*

$$\| \mathbf{G}_\varepsilon^2 - \mathbf{Q}_\varepsilon \|_{\mathscr{L}(L^2(M_\varepsilon), L^2(M_\varepsilon))} \le C\varepsilon^{3-\sigma} |\log \varepsilon|^{1/2} (\varepsilon^{\sigma+\theta} + \varepsilon^{(1/2)+\theta})$$

for $\theta > 0$.

The above Proposition can be obtained by using the following

Lemma 1. *Consider the equation*

$$\Delta v_\varepsilon(x) = 0 \quad x \in M \setminus \bar{B}_\varepsilon$$
$$v_\varepsilon(x) = 0 \quad x \in \partial M$$
$$v_\varepsilon(x) + k\varepsilon^\sigma \frac{\partial}{\partial \nu_x} v_\varepsilon(x) = \alpha(\omega) \quad x = w + \varepsilon\omega \in \partial B_\varepsilon.$$

Here $\omega \in S^3 = \{x ; |x| = 1\}$. Then, v_ε satisfies

$$\| v_\varepsilon \|_{L^2(M_\varepsilon)} \le C\varepsilon^{3-\sigma} |\log \varepsilon|^{1/2} \| \alpha \|_{H^{(1/2)+\tilde{\theta}}(S^3)}$$

for $\tilde{\theta} > 0$, where $H^{(1/2)+\tilde{\theta}}(S^3)$ is the L^2 Sobolev space of fractional order. Here C is a constant independent of ε. We can take $\tilde{\theta}$ as close as 0.

We put

$$p_\varepsilon(x, y) = G(x, y) + g(\varepsilon) G(x, \tilde{w}) G(\tilde{w}, y) + h(\varepsilon) \langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle$$

and

$$(\boldsymbol{P}_\varepsilon f)(x) = \int_{M^\varepsilon} p_\varepsilon(x, y) f(y) dy.$$

Fix $f \in L^2(M_\varepsilon)$. We have $(\boldsymbol{G}_\varepsilon^2 - \boldsymbol{Q}_\varepsilon) f = v_1 + v_2$, where $v_1 = \boldsymbol{G}_\varepsilon(\boldsymbol{G}_\varepsilon - \boldsymbol{P}_\varepsilon) f$ and $v_2 = (\boldsymbol{G}_\varepsilon \boldsymbol{P}_\varepsilon - \boldsymbol{Q}_\varepsilon) f$.

We have the following identities.

$$\begin{aligned} \Delta v_1(x) &= (\boldsymbol{G}_\varepsilon - \boldsymbol{P}_\varepsilon) f(x) \quad && x \in M_\varepsilon \\ v_1(x) &= 0 && x \in \partial M \\ v_1(x) + k\varepsilon^\sigma \frac{\partial}{\partial \nu_x} v_1(x) &= 0 && x \in \partial B_\varepsilon \end{aligned}$$

and

$$\begin{aligned} \Delta v_2(x) &= 0 \quad x \in M_\varepsilon \\ v_2(x) &= 0 \quad x \in \partial M. \end{aligned}$$

Thus, if we estimate

$$v_2(x) + k\varepsilon^\sigma \frac{\partial}{\partial \nu_x} v_2(x)$$

on ∂B_ε, then v_2 can be estimated using Lemma 1.

We stated a rough story of our proof of Proposition 1. If Proposition 1 is proved, the rest part of our proof of Theorem 1 is a perturbation theory of eigenvalues.

Comment. The details of the present paper will appear elsewhere.

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